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JÓZEF MARCINKIEWICZ: ANALYSIS AND PROBABILITY

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Introduction. The life and work of Józef Marcinkiewicz (1910–40), the centenary of whose birth is commemorated in this volume, is a fascinating story for several reasons.

First, there is the quite extraordinary achievement of a precociously talented young mathematician writing 55 papers in the seven years 1933–40, a number of which have become enduring classics, tragically terminated by his untimely and violent death. This story exemplifies, in one life's work, the larger story of the extraordinary flowering of Polish mathematics between the wars, say from the foundation of *Fundamenta Mathematicae* in 1920 to the destructive impact of the tragic events of 1939–40.

Secondly, there is the impact of Marcinkiewicz on analysis, still ongoing, and exemplified in the famous dedication of Zygmund's *Trigonometric Series*: 'Dedicated to the memories of A. Rajchman and J. Marcinkiewicz, my teacher and my pupil'. This is to be seen in the great achievements of the Calderón–Zygmund–Stein school of analysis so influential today. Here one finds such themes as the interplay between real and complex analysis, and between analysis in one and many dimensions.

Thirdly, there is the impact of Marcinkiewicz on probability (of particular interest to me as I am myself a probabilist). Here we find such classic results as the Marcinkiewicz–Zygmund inequalities and strong law of large numbers, and the Marcinkiewicz–Zygmund law of the iterated logarithm. The work of the Polish school of probability between the wars – these two authors, Steinhaus, Kac and others – bears comparison with that of the French school, and is outshone only by work of the Russian (then Soviet) school of Kolmogorov, Khinchin and others.

Fourthly, one sees ever more strongly as time passes the constructive interplay between analysis and probability here, exemplified in such topics as singular integrals, maximal inequalities in analysis and martingale theory, type and cotype in the geometry of Banach

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space, weak-type inequalities in analysis and probability, H^1 and BMO, inequalities of Burkholder–Davis–Gundy type, and wavelets.

1. Józef Marcinkiewicz and Polish mathematics.

Life. The most important source of information about Marcinkiewicz, both personal and mathematical, is his teacher Antoni Zygmund's 28-page appreciation of him [Zyg3] at the beginning of his *Collected Papers* [Mar], which Zygmund edited.

Józef Marcinkiewicz was born on 30 March/12 April 1910 (the change of calendar had not then taken place) in Cimoszka, near Bialystok, Poland. He was a student from 1930–33 at the University of Stefan Batory in Wilno, Poland (now Vilnius, Lithuania). Here his professors of mathematics were Stefan Kempisty (1892–1940 – like Marcinkiewicz, a victim of World War II), Juliusz Rudnicki, and Antoni Zygmund (1900–1992). Zygmund relates how Marcinkiewicz's precocious talents were obvious to his teachers. During 1931–32 Zygmund taught a course on Lebesgue integration and trigonometric series, which he recognized would prove too hard for most students but not for Marcinkiewicz; to Zygmund's relief, Marcinkiewicz took the necessary formal step of asking to attend the course on his own initiative. He took his MA degree in 1933; his MA thesis already contained original results. He did military service in 1933–34, when he became a junior assistant at Wilno. His PhD followed in 1935, under Zygmund; his PhD thesis was an extension of his MA thesis. He spent 1935–36 on a fellowship in Lwów, working with Stefan Kaczmarz (1895–1939) and Juliusz Schauder (1899–1943). Both were victims of the War: Kaczmarz was killed in a train accident on active service; Schauder, a Jew, was a victim of the Holocaust (shot, before the industrialization of the extermination process through gas chambers).

In 1936 Marcinkiewicz became a senior assistant in Wilno, and a docent there in 1937. In the spring of 1939, he held a fellowship in Paris. He was offered a chair by the University of Poznań, where the Józef Marcinkiewicz Centenary Conference very fittingly took place.

In August 1939, as war loomed, he was in England, presumably visiting Littlewood in Cambridge. He was urged to stay in England, but, being an officer in the reserve and a patriot, he returned to Poland. Zygmund recounts seeing him in uniform in Wilno on 2 September. He became a second lieutenant in the 2nd Battalion, 205th Infantry Regiment, and took part in the defence of Lwów, 12–21 September. Following the invasion of Poland by the USSR on 17 September (the Nazi-Soviet Pact, or Molotov–Ribbentrop Pact, was signed on 23 August; the Nazi invasion of Poland started on 1 September; fighting continued to 6 October, though Poland never formally surrendered), Lwów surrendered to the Red (Soviet) Army. Marcinkiewicz became a prisoner of war on 25 September, and was taken for "temporary internment" by USSR. He is known to have been imprisoned in Starobielsk, from where he sent several letters and postcards to his parents, Zygmund and friends, the last in March 1940. The exact date and place of his death are unknown, but he is presumed to have been executed in the Katyń Massacre (there were several execution sites, in Katyń, Starobielsk, Kozielsk and Kharkov). This is commemorated on 10 April (this year's commemoration, to mark the 70th anniversary, was marred by the tragic plane crash which killed the President of Poland and many others). For further information on Marcinkiewicz's life and death see [DaHC], and on the Katyń Massacre, see [SS/G/M][1,2]

Polish mathematics between the Wars. The nineteenth century was the heroic period for mathematics – the bulk of the undergraduate curriculum dates from this time. But the nineteenth century was the period when Poland was absent from the map of Europe, being divided between Austria, Prussia (later Imperial Germany) and Russia. Individual Polish mathematicians did contribute, but their work is usually considered in the context of the relevant national school (one example being the statistician Bortkiewicz, and his classic study on deaths from kicks of a horse in the Prussian cavalry).

During this period, there was a lack both of universities and of opportunities for study in mathematics and science, which naturally hindered mathematical and scientific work. For example, in the Russian-ruled part of Poland, following the suppression of the 1863 Uprising the University of Warsaw was closed as a punishment, being replaced by a Russian-language Imperial University which Poles boycotted, while in the German part there was no university; Polish was banned from high schools in both Russian and German parts. It was natural for Polish intellectuals of the time to focus on the arts and literature, in order to keep the language alive. All these were factors in the relative quiescence of Polish science and mathematics before World War I.

A fine account of the history of Polish mathematics in modern times is given by Kuratowski in his *A half-century of Polish mathematics: Remembrances and reflections* [Kur2] (see in particular the Preface by Ulam). Kuratowski discusses Polish mathematics before and during WWI (Ch. 1), between the Wars (Ch. 2), under Nazi occupation (Ch. 3), and post-war (Ch. 4). One famous incident during WWI took place in a park

[1] This book was published recently in connection with the dedication on 19 September 2009 of the memorial at the National Memorial Arboretum at Alrewas (near Lichfield, Staffordshire) to the Poles who fought in British uniform in WWII. The chapter 'Katyń: Mass murder at the stroke of a pen' (p. 252–262) contains photographs, text, copies of the recommendation from Beria to Stalin and the Politburo that Polish officer prisoners be shot, signatures of assenting Politburo members (Stalin, Voroshilov, Molotov, Mikoyan, Kalinin, Kaganovich), and translations.

[2] Of course, the geopolitical background to these events was completely changed by Operation Barbarossa, the invasion by Nazi Germany of the Soviet Union on 22 June 1941. Immediately Stalin became of necessity a wartime ally of the UK. The Katyń Massacre hung over relations between the UK Government and the Polish Government-in-Exile in London under General Sikorski. Churchill wrote in his war memoirs *The Second World War* 'It was decided by the victorious Governments concerned that the issue should be avoided and the crime of Katyń was never probed in detail'. These matters are in the public domain, but not much discussed. In one recent life of Churchill, by the British politician Roy Jenkins [Jen], the name Katyń does not appear.

Jenkins is, however, explicit on the excruciatingly embarrassing way in which Britain's policy towards Poland changed from going to war on her behalf in September 1939, to agreeing with Stalin to shift her borders westwards, without consulting her, at the Tehran Conference of November 1943 (Marcinkiewicz's Wilno became Vilnius, in Lithuania from 1940–41 and 1990 on, the Lithuanian SSR, USSR, 1941–1990), to the 'naughty document' he signed with Stalin in October 1944, dividing up spheres of influence in Europe.

in Krakow, where Steinhaus overheard two young men discussing the Lebesgue integral – one was Banach, the other was Nikodym ([JakMir], 1.2).

The founding of the journal *Fundamenta Mathematicae* in 1920 is one seminal event in Polish mathematics between the Wars (the founding editor was Janiszewski, but he died before the first issue appeared; he was succeeded by Mazurkiewicz and Sierpiński). The idea here was to provide a forum for Polish mathematics on the world stage, and so to be written in the major western languages. The concentration on foundational questions – set theory, mathematical logic, topology – was partly a result of the impact of Cantor's work in the last quarter of the nineteenth century. These areas were topical and promising, but also so young that Polish mathematicians could enter a relatively new field, rather than a more established one where their lack of a national mathematical tradition would put them at a disadvantage.

Dramatic evidence of the success of this launch of the new Polish mathematics was provided by the launch a decade later of the series *Monografje Matematyczne* in 1932. We note the first seven volumes:

- I, *Théorie des opérations linéaires* by Stefan Banach in 1932 [Ban];
- II, *Théorie de l'intégrale* by Stanislaw Saks in 1933 (second edition, *Theory of the integral* in 1937, VII [Sak]);
- III, *Topologie I* by Kazimierz (C.) Kuratowski in 1933 [Kur1];
- IV, *Hypothèse du continu* by Waclaw Sierpiński in 1934 [Sie];
- V, *Trigonometrical series* by Antoni Zygmund in 1935 [Zyg1];
- VI, *Theorie der Orthogonalreihen* by Stefan Kaczmarz and Hugo Steinhaus in 1935 [KaSt].

It would be difficult to match the superb quality of this in the opening volumes of any series in the whole of mathematics.

Further background on Polish mathematics just before WWII is given in the admirably written autobiography of Mark Kac (1914–84), a pupil of Hugo Steinhaus (1887–1972) in Lwów [Kac2].

Holgate [Hol] gives an account of *Independent functions: Probability and analysis in Poland between the Wars*; we shall return to this, and its sequel [Bin3], later.

Marcinkiewicz's work. In what follows, we briefly review Marcinkiewicz's work, on analysis, on probability, and on the interplay between the two. Our emphasis is on the continuing vitality of Marcinkiewicz's work, as evidenced by its influence on the standard works. What is striking is how many of the themes that Marcinkiewicz studied (alone, or with Zygmund) are very much alive today. What this demonstrates is that Marcinkiewicz and Zygmund, as well as having extraordinary mathematical ability, also had excellent mathematical taste.

Marcinkiewicz wrote 55 papers, appearing in 1933–45 (the last few posthumous); 15 were in English, 40 in French (the author can confirm that Marcinkiewicz wrote beautiful English, and he has it on the authority of French colleagues that the same is true of his French). He collaborated with Zygmund (15), S. Bergman (2), B. Jessen, S. Kaczmarz, R. Salem. Zygmund analyzed his papers under the following headings:

functions of a real variable; trigonometric series; trigonometric interpolation; functional operations; orthogonal systems; functions of a complex variable; calculus of probability.

2. Analysis. We will refer to Marcinkiewicz's papers by using the 'square-bracket number' – their number in the Marcinkiewicz bibliography on p. 31–33 of his *Collected Papers*, [Mar].

The first mention of Marcinkiewicz in the textbook literature is his two citations in [Zyg1], when he was 25. The first is of his first paper [1], a one-page proof of a result of Kolmogorov (*Fundamenta Mathematicae*, 1924) on a.e. convergence of lacunary Fourier series. The second [6] is on Riemann's two methods of summability (this is cited in French, but published in English).

In the much later [Zyg2], which Zygmund always regarded as the second edition of [Zyg1] (the books are always referred to in English as the 'little Zyg' and the 'big Zyg'), Marcinkiewicz is cited twelve times. We will refer to these by section number in [Zyg2], and by square-bracket number as above.

IV.2; [8], [28], [32] ('Theorem of Marcinkiewicz' in Zygmund, usually referred to as Marcinkiewicz's integral): for F closed in (a,b), $f \in L_1(a,b)$, $\chi(.)$ the distance from F, $\lambda > 0$, its λth power χ^λ satisfies

$$J_\lambda(x) := \int_a^b f(t)\chi^\lambda(t)dt/|t-x|^{\lambda+1} < \infty$$

a.e. on F, and

$$\int_F |J_\lambda| \le (2/\lambda) \int_a^b |f|.$$

The Marcinkiewicz integral is central to the Calderón–Zygmund proof of the weak-type $(1,1)$ estimate for singular integrals in higher dimensions ([CaZy]; [Ste3], Ch. 1). In view of the importance of the Calderón–Zygmund theory of singular integrals (to which we return below), this may be considered one of Marcinkiewicz's most influential contributions to analysis (together with his interpolation theorem, below).

VIII.3; [8]: there exists an integrable function the partial sums of whose Fourier series are a.e. bounded and divergent.

IX, Miscellaneous theorems and examples 8 [6] (Riemann's two methods of summation are not comparable) and 16 [18] (with Zygmund) (on sets of uniqueness for trigonometric series).

XII.4 [44]: Marcinkiewicz interpolation theorem. This was a brief note of 1939 in the *Comptes Rendus*, published without proof; the proof was later given by Zygmund in 1956 [Zyg4]. This gives an operator of strong type interpolated between two operators of weak type (we refer to e.g. [Zyg3] and [Bas] V for definitions here). This has proved one of Marcinkiewicz's most influential contributions. It is often referred to as the 'real method' of interpolation, in contrast to the Riesz–Thorin 'complex method' (see e.g. Grafakos [Gra1], 1.3).

XII, Miscellaneous theorems and examples 1–4 [15] (with Zygmund): orthogonal series.

XIII.3 [5]: if $f \in L_1$ satisfies

$$\frac{1}{h}\int_0^h |f(x+t) - f(x)|dt = O(1/\log 1/|h|)$$

as $h \to 0$ for all $x \in E$, then the Fourier series of f converges a.e. on E.
XIII, Miscellaneous theorems and examples 8 [11]: interpolation.
XIV.5 [32]: the Marcinkiewicz function μ. This is analogous to the Littlewood–Paley function g (and as we shall see, both are analogous to the quadratic variation of a martingale), and also to the Lusin area integral.
XV.4 [40]: multipliers of Fourier series. This has been a very influential paper. It is closely linked to Littlewood–Paley theory (for which see [Zyg2] XV and e.g. [Ste2], [EdGa]).
XVII.2 [7]: strong differentiability of multiple integrals and its applications.
XVII.3 [43], [46]: summability of double Fourier series.
XVII.4 [55]: analytic functions of two variables.

Zygmund's most important collaboration in his Polish period before WWII was with Marcinkiewicz; his most important collaboration in his US period after WWII was with Alberto Calderón (1920–98) (see [Zyg4] for all twelve Calderón–Zygmund papers, [Cal] for five of them, with commentary in both cases). The most distinguished product of the Calderón–Zygmund school is E. M. Stein. Stein's paper [Ste1], on the Littlewood–Paley, Lusin and Marcinkiewicz functions, brings together the quadratic functions above (for Littlewood–Paley theory, see [Ste2]). His book [Ste3] cites the Marcinkiewicz integral (I.2.3), the Marcinkiewicz interpolation theorem (I.4.1 and Appendix B), and the Markinciewicz multiplier theorem (IV.3, in the context of Littlewood–Paley theory and multipliers, IV). His later book [Ste4] cites the multi-dimensional maximal function (II.5.E – the paper [7] of Marcinkiewicz, Jessen and Zygmund). The book by Stein and Weiss [SteWei3] also cites the Marcinkiewicz multiplier theorem (IV.2). Grafakos [Gra1] cites the Marcinkiewicz–Zygmund inequality [45] for linear operators, the Marcinkiewicz function (4.6.6) (see also [Gra2], 6.4.5), the Marcinkiewicz interpolation theorem (1.3) and the Marcinkiewicz multiplier theorem, in one (5.2.1) and n (5.2.2) dimensions. See also Weisz [Wei2].

Meyer's book on wavelets refers ([Mey], 6.2) to the Marcinkiewicz interpolation theorem. It refers forward to the sequel with Coifman ([MeyCoi], 7.2). This book begins ([MeyCoi], 7.1) with a historical account of the evolution of Calderón–Zygmund operators (citing four of their papers, over the period 1952–57). One of the key developments here new since Marcinkiewicz's time was of course the Schwartz theory of distributions or generalized functions.

Maximal inequalities. The Hardy–Littlewood paper of 1930 on maximal inequalities ([HL1]; [HLP], 10.18) is one of the papers Marcinkiewicz cited most often, and maximal inequalities run right through his work, alone and with Zygmund. This theme is continued in the Calderón–Zygmund collaboration (1950 on), in Stein's work (above), and in probability (Sections 3 and 4 below – see e.g. the work of Burkholder and Gundy [BG]; [Bur4]).

There are many situations in which maximal inequalities are not only sufficient, but also necessary. This line of thought goes back to Kolmogorov in 1925, Stein in 1961, Burkholder in 1964, Sawyer in 1966 and Garsia in 1970; see Krengel [Kre], Notes to Ch.

1 for details and references.

Interpolation spaces. Marcinkiewicz's work on interpolation is discussed by Zygmund in his commentary [Zyg3]. As an indication of the influence of the Marcinkiewicz interpolation theorem, we mention the Burkholder–Gundy paper [BG], and the extensive textbook literature that has grown up on interpolation spaces: Bennett and Sharpley [BeSh], Bergh and Löfström [BeLo], Triebel [Tri], and Krein, Petunin and Semenov [KPS]. In particular, the work on maximal inequalities mentioned above leads to the idea of rearrangement-invariant spaces (for which see e.g. [LiTz], II Ch. 2), and the Marcinkiewicz spaces M_ϕ mentioned by several speakers at this conference.

Hardy spaces. The Marcinkiewicz–Zygmund paper [31], 'A theorem of Lusin', is on the Lusin area integral and Hardy spaces. The subject of Hardy spaces (for which see e.g. Hardy's *Works* III.2 [Har2] and Duren [Dur]) is vast and important; we shall return to such matters in Section 4.

Higher dimensions: real methods. The paper [7] by Marcinkiewicz, Jessen and Zygmund deals with differentiability of multiple integrals. It uses the concept of strong differentiability, and the condition (in R^n) $|f|(\log_+|f|)^{k-1}) \in L$.

Higher dimensions: complex methods. Traditionally, one of the limitations of complex methods has been that they are regarded as restricted to one (complex) variable; the theory of functions of several complex variables, though very interesting, is very different from the one-variable case and much harder. Again traditionally, there have been three main approaches to complex analysis: via complex differentiability (Cauchy), power series (Weierstrass), and the Cauchy–Riemann equations. Stein and Weiss [SteWei2] realized that several complex variables was the wrong setting in which to seek generalizations of their one-dimensional results. Instead, they pioneered Cauchy–Riemann systems of equations in n variables, using Hardy spaces $H^p(R^n_+)$ in place of $L^p(R^n)$. For background, see Fefferman [Fef], 10.

Convergence of Fourier series. Perhaps the theorem that Marcinkiewicz might most have liked to see is the Carleson–Hunt theorem on convergence of Fourier series (Lennart Carleson in 1966 [Car], R. A. Hunt in 1968 [Hun]). Unfortunately this did not appear in time to be included in [Zyg2], for long the standard work; see e.g. [Gra2] Ch. 11 for a textbook account.

Gap theorems. Marcinkiewicz wrote several papers on lacunary (gap) series: [1] and [30] (with Zygmund) on Fourier series, [29] on orthogonal series. The subject was being worked on independently in the US at the same time; for a textbook account, see the book [Lev] by Norman Levinson, a pupil of Wiener.

Non-absolute integrals. The Lebesgue integral is absolute (f is integrable iff $|f|$ is), and one needs non-absolute integrals for Fourier series, and to link differentiation more closely with integration, both favourite themes of Marcinkiewicz. For the Denjoy and Perron integrals, see [Zyg2] XI.6, [Sak] VIII. For the Burkill integral, see H. R. Pitt's obituary of J. C. Burkill [Pit]. For the Henstock and Kurzweil integrals, see e.g. McLeod [McL].

Singular integrals. The main thread running through the work of the Calderón–Zygmund–Stein school of analysis is singular integrals, in particular the Hilbert and Riesz transforms. Their relevance to analysis is well expounded in the books [Ste3], [Ste4], [SteWei3] cited above. For their relevance to probability, and in particular to martingale theory, see [Bur5], [Bas] IV and Section 4 below.

Paley. . The life of the British analyst R. E. A. C. Paley (1907–33) bears striking similarities with Marcinkiewicz's in several ways: both were brilliant, died young, were interested primarily in analysis, also worked in probability, and collaborated with Zygmund. For background on Paley's life and death, see Hardy [Har1], Wiener [Wie].

The collaboration of Paley, Wiener and Zygmund [PWZ] of 1933 – the subject of the last two chapters in the book [PaWi] by Paley and Wiener of 1934 – concerns the expansion of Brownian motion into a random series with respect to a suitable orthonormal system with independent Gaussian coefficients (cf. [Kah1]). We return to this important matter later.

Orthogonal series. In 1932, Paley [Pal] gave for $p > 1$ the inequality between the pth moments of a function f and its square function $S(f)$, and between moments of f and f^*, its maximal function with respect to the Walsh martingale (in the language of Sections 3 and 4 – generated by the Rademacher functions [Kah1]). Marcinkiewicz [20] obtains analogous results for the Haar functions. He studies orthogonal series in several papers, [10], [15], [20], [21], [29]; see the commentary in [Zyg3], 21–23. We note that 2010 marks also the centenary of Walsh's paper [Wal], cited in [20].

3. Probability. The Kolmogorov axiomatics of the *Grundbegriffe der Wahrscheinlichkeitsrechnung* of 1933 [Kol] were still quite recent. We speak naturally nowadays of *independent random variables*, taking 'random variable' as 'measurable function'. The Marcinkiewicz–Zygmund papers [16] and [25] of 1937 and 1938, and the Marcinkiewicz papers [22], [23] and [27] of 1938, speak of *independent functions*: 'Les résultats obtenus peuvent être traduits en langage de la théorie des probabilités, ce que nous laissons au lecteur' ([22]; [Mar] 328). For background to this Polish work, see Kac's autobiography [Kac2], his lovely book [Kac1], Holgate's paper [Hol], and its sequel, [Bin4].

We begin with random series – series of the form ΣX_n with X_n random variables, independent in the first instance. This area goes back to Kolmogorov's three-series theorem of 1928 (and indeed to Khinchin and Kolmogorov in 1925 – see e.g. [Bin1] Section 3 for details). The area was taken further in a series of notes by Paley and Zygmund in 1930–32 [PaZy]. The Paley–Zygmund work inspired the modern classic monograph on the subject, by Kahane ([Kah1], xi and Sections 3.3, 5.5). It also led to Marcinkiewicz's best-known work in probability, his 1937 paper [16] (cf. also [25] in 1938) with Zygmund, discussed below.

In [16], the authors link, for independent X_n with mean 0 and variance 1, convergence of $\Sigma a_n X_n$ a.s. with convergence of Σa_n^2. This led to a martingale generalization by Gundy [Gun2], and to the Burkholder–Gundy paper [BG]; see Section 4 below. In [25], Th. 8, the authors show the equivalence of convergence in probability and almost surely of random series of independent terms (a result that they note in proof was obtained the previous year, 1937, by Lévy in [Lév1], Th. 44 p. 139). This result extends to general Banach spaces (indeed, with convergence in distribution added to the equivalences); see Itô and Nisio [ItNi].

A convenient modern textbook source for the Marcinkiewicz–Zygmund work on probability is the book by Chow and Teicher [ChTe], where we find:
10.3: *Marcinkiewicz–Zygmund inequality* [16]. For $p \geq 1$, there are constants A_p, B_p such that for X_n independent, 0-mean random variables,

$$A_p \|(\Sigma_1^n X_j^2)^{1/2}\|_p \leq \|\Sigma_1^n X_j\|_p \leq B_p \|(\Sigma_1^n X_j^2)^{1/2}\|_p.$$

For an extension from powers x^p to more general functions $\Phi(x)$, see [BG], Cor. 5.4.
5.2: *Marcinkiewicz–Zygmund Law of Large Numbers* (LLN). For X, X_n independent and identically distributed (iid), $S_n := \sum_i^n X_i$, and $0 < p < 2$, the following are equivalent:

(i) $X \in L_p$, i.e. $E|X|^p < \infty$;
(ii) there exists a constant c with

$$(S_n - nc)/n^{1/p} \to 0 \quad a.s. \quad (n \to \infty)$$

(and then w.l.o.g. $c = EX$ if $1 \leq p < 2$, while c is arbitrary if $0 < p < 1$).

This important result – which provides an extension of the Kolmogorov strong law of large numbers (the $p = 1$ case) – may be generalized in various directions. For a range of alternative extensions to the Kolmogorov strong law, see [Bin1]. For ϕ-mixing, see Peligrad [Pel]; for α-mixing see Rio [Rio]. For Banach spaces, see below.

Marcinkiewicz–Zygmund Law of the Iterated Logarithm (*LIL*). According to Kolmogorov's LLN, if X_n have mean 0 and finite variances, $S_n := \sum_1^n X_k$, $s_n^2 := var(S_n) \to \infty$, then if

$$|X_n| = o(s_n/\sqrt{\log\ \log\ s_n}) \quad a.s.$$

then

$$\limsup\ S_n/\sqrt{2s_n \log\ \log\ s_n} = +1 \quad a.s.$$

Marcinkiewicz and Zygmund [19] showed that this is sharp: one cannot replace o here by O. For a textbook reference, see e.g. Stout [Sto], 5.2.

Maximal inequalities. Kolmogorov used maximal inequalities (before Hardy and Littlewood [HL] in 1930!) to prove strong (a.s.) limit theorems. For details, see e.g. [Bin2], [Bin3]. Use of maximal inequalities for martingales (below) was pioneered by Doob [Doo], VII.3.

Martingales. One can generalize beyond the sum of independent 0-mean random variables to martingales. The inequalities of Paley [Pal] for the Walsh case, Marcinkiewicz [20] for the Haar case and Marcinkiewicz–Zygmund [16] for sums of independent random variables (all known by 1937) eventually become unified in the *Burkholder–Davis–Gundy* (BDG) inequalities [BDG] of 1972; for textbook accounts see e.g. [Kal], Prop. 15.7, Th. 23.12, [Gar].

The importance of Marcinkiewicz's work on martingales is developed in many papers by D. L. Burkholder and R. F. Gundy from 1966 on; see in particular [BG], [BDG] (with Burkholder's pupil B. J. Davis), and [BGS], with Silverstein. I have heard Burkholder speak many times, always mentioning the name Marcinkiewicz (which I first heard from him). *Probability in Banach spaces*

The ideas of *type p* and *cotype p* have proved useful in the geometry of Banach spaces. It was shown by A. de Acosta [Aco] that for a Banach space B and $p \in [1, 2)$, the following are equivalent:

(i) the Marcinkiewicz–Zygmund LLN holds for B;
(ii) B has type p.

See e.g. Ledoux and Talagrand [LeTa], 7.2, 9.3. We return to the geometry of Banach spaces in Section 4.

Infinite divisibility [22], [23], [27]. These papers relate to the theory of infinitely divisible random variables, nowadays studied in tandem with Lévy processes. The textbook account of this area most relevant to Marcinkiewicz's time is Gnedenko and Kolmogorov [GnKo].

Analytic characteristic functions [27], [35]. Paper [27] above contains some complex-analytic results. Paper [35] contains the following striking theorem: for P_m a polynomial of degree $m > 2$, the function $\exp\{P_m(.)\}$ is not a characteristic function (Fourier–Stieltjes transform of a probability distribution). For a textbook account of complex methods for characteristic functions, see Lukacs [Luk].

Brownian motion and harnesses [41]. This interesting paper (on Brownian motion – the name harness came later) was the subject of a talk at this conference by Kwapień, who pointed out that Zygmund does not mention it in [Zyg3], nor has it been cited since. We will refer to his contribution to this volume for details. We note that harnesses have been studied recently in [BMW1], [BMW2].

Döblin. As with Paley in Section 2, there are uncanny parallels between the lives of Marcinkiewicz and the probabilist Wolfgang Döblin (Doeblin, 17.3.1915–21.6.1940). Born in Berlin as a German Jew (and son of the prominent writer Alfred Döblin), Döblin left Germany in 1933 and settled in Paris, where he studied under Lévy and Fréchet. He wrote his first paper (with Lévy) in 1936, and took his doctorate in 1938. He enlisted in the French Army, and fought until days before the French surrender, preferring to die by his own hand than be captured. Döblin worked mainly on Markov chains and processes. His 26 papers and notes (1937–40) – in which he pioneers coupling – put him in the first rank of 1930s probabilists, but the opening in 2000 of the *pli cacheté* which he deposited with the Académie des Sciences showed how far in advance of his time he was. For mathematical background, see Lindvall [Lin], Bru and Yor [BruYor], and for a biography, Petit [Pet].

4. Analysis and Probability. We turn now to a combination of the themes of the last two sections – analysis and probability interacting. Marcinkiewicz's work begins in analysis, and develops naturally into probability. His use of maximal inequalities in analysis has inspired much work in probability, e.g. the work of Burkholder and Gundy on martingales.

We begin with Kahane's masterly survey [Kah2] (which contains many references). This refers to the Paley–Wiener–Zygmund paper of 1933 [PWZ], on random series (cf. [PaWi]). Here one finds Brownian motion represented as the sum of random series of various kinds – e.g. Rademacher series (Paley–Zygmund) and Fourier series (Wiener). Lévy's 'broken-line' construction of Brownian motion (in his book of 1948, [Lév2]) can be seen to give a similar series expansion of Brownian motion, using the Schauder functions). We now use *wavelet* expansions [KaLR]. For a textbook account, see e.g. Steele [Ste], 3.4. One can use any complete orthonormal series on any Banach space (Itô and Nisio [ItNi], Section 5).

Lévy [Lév2] also links Brownian motion with analytic functions: with f analytic and $Z = (Z_t)$ complex Brownian motion ($Z = X + iY$ with X, Y independent linear Brownian motions), $f(Z_t)$ is again complex Brownian motion but moving with variable speed: $f(Z_t)$ is $Z^*(T_t)$, with Z^* Brownian motion and $T_t = \int_0^t |f'(Z_u)|^2 du$.

The potential for cross-fertilization between analysis and probability is clearly prefigured in the work of Marcinkiewicz and of Marcinkiewicz–Zygmund reviewed above. The famous chapter on martingales in Doob's book [Doo] of 1953 led to a surge of interest in martingales, both in order to extend results on sums of centred independent random variables to martingales and to make links with analysis. Three key papers here, all from 1966, are Austin's result that L_1-bounded martingales have finite quadratic variation [Aus], Burkholder's work on martingale transforms [Bur1], and Gundy's link between martingales and convergence of orthogonal series [Gun1] (Burkholder credits Gundy with the crucial steps here – [Bur1], 1499 and 1502).

As mentioned at the end of Section 2, the two-sided inequalities linking the p-norms of f and $S(f)$ for Walsh series ([Pal], 1932) and Haar series (Marcinkiewicz [20], 1937) hold for $p > 1$. A similar two-sided inequality linking the p-norms of f and f^* follows from the Hardy–Littlewood maximal inequality. These do not extend to $p \in (0, \infty)$, but they lead to a two-sided inequality linking the p-norms of $S(f)$ and f^*. This last inequality *does* extend to all $p \in (0, \infty)$, by interpolation. Indeed, the inequality for *two* such p implies the inequality for *all* such p. This is the starting point of the important paper [BG]. This also contains analogous results linking the p-norms of a stopping time τ and that of B^*_τ, the maximal function of Brownian motion stopped at τ, again for all $p \in (0, \infty)$. The extension (or extrapolation) to $p \in (0, 1)$ is particularly interesting because, while L^p is a Banach space for $p \geq 1$, it is only a quasi-Banach space for $p \in (0, 1)$.

The link between Brownian motion and Hardy spaces is explored by Burkholder, Gundy and Silverstein [BGS], who obtained a characterization of Hardy spaces by maximal functions. See [Bur4], [Bur5], [Gun3], [Gun 4] ([Gun3] contains valuable historical remarks on p. 255–264, linking Marcinkiewicz's work of the 1930s via Zygmund to the work of Calderón in the 1950s and Calderón and Zygmund in the 1960s on singular integrals, and Fefferman and Stein in the 1970s on Hardy spaces; it also contains many references). Davis ([Dav1], [Dav2]) developed Lévy's link between Brownian motion and analytic functions, in particular obtaining a proof of Picard's (little) theorem in this way. Textbook accounts of such inter-relations are Durrett's book [Durr] of 1984 and Bass's book [Bas] of 1995. Both deal with the duality between H^1 and BMO (Fefferman's theorem, and the Fefferman–Stein decomposition), and with boundary behaviour of analytic functions (in the unit disk, say). To give one specific result: the Marcinkiewicz–Zygmund paper [31] of 1938 on Lusin's theorem is completed by a 1943 result of Spencer [Spe] (linking finiteness of the area integral with non-tangential boundedness; Spencer's theorem was extended to higher dimensions by Calderón in 1950 and Stein in 1961). The probabilistic approach not only provides alternative (and sometimes simpler) proofs of the analytic results, it also makes them transparent. We refer to [Durr] Ch. 4 for details here, and to Weisz [Wei1] for more on Hardy spaces and martingales; see also [KaLR] Part I, Ch. 11.

Martingales and the geometry of Banach space. We have already encountered one link between probability and the geometry of Banach space – de Acosta's result linking type p with the Marcinkiewicz–Zygmund LLN for pth moments. Another is Chatterji's theorem [Cha]: the Radon–Nikodym theorem and the martingale convergence theorem are the *same theorem*, in that if one holds for a Banach space B, so does the other. A third is Burkholder's result [Bur3], giving a geometrical condition for the UMD property (martingale differences being unconditional) – see also [Bur4], and for an overview of the area, [Bur5]. These are also the spaces where the Hilbert transform is continuous on L^2 (Bourgain [Bou]; [MeyCoi], 42).

For background to this important area, we refer to [LeTa], or the more recent survey by Maurey [Mau].

Martingales and differentiation. Marcinkiewicz was interested in differentiation – e.g. in [7] with Jessen and Zygmund. Martingale theory can be systematically applied here too; see Hayes and Pauc [HaPa].

Hardy spaces on polydiscs. For the extension of Hardy spaces from the disc to the polydisc (from one to several complex variables), we refer to Malliavin and Malliavin [MaMa], Gundy and Stein [GunSte].

Hardy. Hardy was a pure mathematician notoriously uninterested in applied mathematics, and in his formative years probability theory had not yet come of age. Nevertheless, the prominence of both Hardy spaces and probability in this paper suggests hidden links. These are indeed there, and are explored in an intriguing paper by Diaconis [Dia]. In particular, he quotes Erdös' astute comment that if Hardy had known any probability at all he would have discovered the law of the iterated logarithm.

Rajchman measures. We close with a brief mention of the mathematics of Zygmund's teacher Aleksander Rajchman (1890–1940 – arrested by the Gestapo April 1940; died, Sachsenhausen concentration camp, probably July or August 1940). *Rajchman measures* are probability measures on the unit circle whose Fourier coefficients tend to zero. They have been characterized by Russell Lyons [Ly]; Rajchman measures are important in OPUC, the theory of orthogonal polynomials on the unit circle.

Postscript. The mathematical lives of Marcinkiewicz and Zygmund were deeply intertwined. Their personal lives were different (apart from in their lengths, 1910–40 and 1900–92), as the following thought and anecdote illustrate.

A thought, concerning Marcinkiewicz: if only he had been physically unable to return to Poland before war broke out, he might well not only have survived, but been able to fight with the Polish forces in British Army uniform. For an account of the splendid performance of the Polish I and II Corps, under Generals Maczek and Anders, see [SS/G/M] or any standard account of WWII (Maczek took part in the defence of Lwów, as did Marcinkiewicz). The author, a Briton and a war baby (b. 1945), had for many years a daily reminder of the Polish forces (I drove to work past the Polish War Memorial (Air Force) at Northolt, Middlesex).

An anecdote, concerning Zygmund: he too was an officer in the reserve. Following the end of the fighting, the Soviet occupation authorities ordered officers to report and register. Zygmund went along to the designated office to register. He was brusquely told to go away by the soldier on duty. Zygmund persisted. The soldier (no doubt all too well aware of the fate awaiting those who registered) then told Zygmund to go away in language of such brutal obscenity that Zygmund took this life-saving hint, went away, left Poland (though the country did not formally exist at that time) and settled in the US. Here in the rest of his long life he greatly enriched our subject, founded the Calderón–Zygmund–Stein school of analysis, and by his work and editorship of Marcinkiewicz's *Collected Papers* ensured that Marcinkiewicz's life and work would be given the recognition they deserve.

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