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BANACH CENTER PUBLICATIONS, VOLUME **
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 201*

INTERPOLATION OF QUASICONTINUOUS FUNCTIONS

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Abstract. If C is a capacity on a measurable space, we prove that the restriction of the K-functional $K(t, f; L^p(C), L^\infty(C))$ to quasicontinuous functions $f \in QC$ is equivalent to

$$K(t, f; L^p(C) \cap QC, L^\infty(C) \cap QC).$$

We apply this result to identify the interpolation space $(L^{p_0,q_0}(C) \cap QC, L^{p_1,q_1}(C) \cap QC)_{\theta,q}$.

1. Introducction. Recently, the concept of capacity has become a tool that is used in much the same way as measure is used. The way to integrate with respect to a capacity C, which is not necessarily an additive set function, is to define the integral of a nonnegative function f using the distributional form of a Lebesgue integral, as proposed by Choquet in his seminal work on capacities [9]:

$$\int f\, dC := \int_0^\infty C\{f > t\}\, dt.$$

With this definition, the Lebesgue capacitary spaces can be introduced and they still have many of the properties of the usual Lebesgue spaces if the capacity $C\colon \Sigma \to [0, \infty]$, defined on a σ-algebra Σ on Ω, that in our case will be a subset of $\mathbf{R}^n$, has the following properties:

2010 *Mathematics Subject Classification*: Primary 46E30; Secondary 46B70, 46M35, 28A12.
Key words and phrases: Capacity, Lorentz spaces, interpolation
The paper is in final form and no version of it will be published elsewhere.

(a) $C(\emptyset) = 0$,
(b) $0 \le C(A) \le \infty$,
(c) $C(A) \le C(B)$ if $A \subset B$,
(d) $C(A \cup B) \le c(C(A) + C(B))$, where $c \ge 1$ is a constant (quasi-subadditivity), and
(e) $C(A_n) \to C(A)$ whenever $A_n \uparrow A$ (Fatou property).

From now on, we will suppose that C satisfies at least all these properties.

This is the case of the variational capacities, and of the Fuglede [10] and Meyers [12] capacities of nonlinear potential theory. Although they are not Caratheodory metric outer measures, they satisfy a Fatou type condition and, by a general theorem due to G. Choquet (cf. [9]), every Borel set $B \subset \mathbf{R}^n$ is capacitable, this meaning that

$$\sup\{C(K);\, K \subset B,\, K \text{ compact}\} = C(B) = \inf\{C(G);\, G \supset B,\, G \text{ open}\}.$$

Then the σ-algebra of all Borel sets turns out to be a convenient domain for all of them. We refer to [1] and [11] for an extended overview of these capacities.

Another well known class of capacities is that of Hausdorff contents. If h is a continuous increasing function on $[0,\infty)$ vanishing only at 0, which is called a measure function in [5], let μ_h be the corresponding Hausdorff measure on $\mathbf{R}^n$, and let I or I_k represent a general cube in $\mathbf{R}^n$ with its sides parallel to the axes. The use of the corresponding Hausdorff capacity or Hausdorff content,

$$E_h(A) := \inf_{A \subset \bigcup_{k=1}^{\infty} I_k} \Big\{ \sum_{k=1}^{\infty} h(|I_k|) \Big\},$$

is often more convenient than μ_h, and $E_h(A) = 0$ if and only if $\mu_h(A) = 0$.

These capacities of potential theory are very useful to obtain bounds for some classical operators and in this note we are interested in obtaining results on interpolation of capacitary function spaces on $\mathbf{R}^n$ of quasicontinuous functions, starting from previous results on general functions contained in [7], [6] and [8]. This means to obtain a result about restriction of interpolation to the subspace QC of quasicontinuous functions (for the definition, see Section 3).

Our goal is to prove that the restriction of the K-functional of the couple $(L^p(C), L^\infty(C))$ to quasicontinuous functions $f \in QC$ is equivalent to

$$K(t, f; L^p(C) \cap QC, L^\infty(C) \cap QC).$$

Then we will apply this result to identify the interpolation space of the couple of capacitary Lorentz spaces $(L^{p_0,q_0}(C) \cap QC, L^{p_1,q_1}(C) \cap QC)_{\theta,q}$.

The notation $A \lesssim B$ means that $A \le \gamma B$ for some absolute constant $\gamma \ge 1$, and $A \simeq B$ if $A \lesssim B \lesssim A$. We refer to [2] for general facts concerning function spaces.

2. Definitions. Under the above mentioned properties on C, a capacity on Σ, we will say that (Ω, Σ, C) is a capacity space. In the theory of capacitary function spaces, it plays the role of a measure space (Ω, Σ, μ) in the theory of Banach function spaces. In [8] we describe the properties for measure spaces that are still satisfied by the capacity spaces.

We consider two functions, f and g, equivalent if they are equal C-q.e. In this case $\int |f|\, dC = \int |g|\, dC$, since $C\{|f| > t\} = C\{|g| > t\}$ for every $t \ge 0$. Thus, $\int |f|\, dC = 0$ if and only if $f = 0$ C-q.e.

The spaces $L^{p,q}(C)$ $(p, q > 0)$ are defined by the condition

$$\|f\|_{L^{p,q}(C)} := \left(q\int_0^\infty t^{q-1}C\{|f|>t\}^{q/p}dt\right)^{1/q} < \infty$$

if $q < \infty$. If $q = \infty$, $\|f\|_{L^{p,\infty}(C)} := \sup_{t>0} tC\{|f| > t\}^{1/p}$.

Observe that $\|f\|_{L^{p,q}(C)} = 0$ if and only if $f = 0$ C-q.e. and equivalent functions have the same $\|\cdot\|_{L^{p,q}(C)}$-norm. Moreover $\|\lambda f\|_{L^{p,q}(C)} = |\lambda|\|f\|_{L^{p,q}(C)}$ and $\|f + g\|_{L^{p,q}(C)} \leq 2c(\|f\|_{L^{p,q}(C)} + \|g\|_{L^{p,q}(C)})$.

We write $L^p(C) = L^{p,p}(C)$ if $p < \infty$ with $\|f\|_{L^p(C)} = \left(\int_\Omega |f|^p dC\right)^{1/p}$. $L^\infty(C)$ is defined as usually by the condition

$$\|f\|_\infty := \inf\{M > 0;\ |f| \leq M\ C\text{-q.e.}\} < \infty.$$

As shown in [8], these spaces are quasi-Banach spaces.

If $\bar{A} = (A_0, A_1)$ is a couple of quasi-Banach spaces, $0 < \theta < 1$ and $0 < q \leq \infty$, the interpolation space $\bar{A}_{\theta,q}$ is the quasi-Banach space of all $f \in A_0 + A_1$ such that

$$\|f\|_{\theta,q} := \left(\int_0^\infty (t^{-\theta}K(t, f; \bar{A}))^q \frac{dt}{t}\right)^{1/q} < \infty$$

where $K(t, f; \bar{A})$ is the K-functional,

$$K(t, f; \bar{A}) := \inf\left\{\|f_0\|_{A_0} + t\|f_1\|_{A_1};\ f = f_0 + f_1\right\}.$$

We refer to [3] and [4] for general facts concerning interpolation theory.

The following results are proved in [8]:

Theorem 1. *Suppose* $0 < \theta < 1$, $0 < p_0 < q \leq \infty$ *or* $0 < p_0 \leq q < \infty$, *and* $\frac{1}{p} = \frac{1-\theta}{p_0}$. *Then*

$$(L^{p_0}(C), L^\infty(C))_{\theta,q} = L^{p,q}(C).$$

Corollary 1. *Let* $0 < p_0, p_1, q_0, q_1 < \infty$, $p_0 \neq p_1$ *and* $0 < \eta < 1$. *Then*

$$(L^{p_0,q_0}(C), L^{p_1,q_1}(C))_{\eta,q} = L^{p,q}(C)$$

with $1/p = (1-\eta)/p_0 + \eta/p_1$.

3. Interpolation of quasicontinuous functions.

Definition 1. *Let* C *a capacity on* $\mathbf{R}^n$. *A function* f *on* $\mathbf{R}^n$, *or on some open subset, is said to be* $C-$*quasicontinuous if for every* $\epsilon > 0$ *there is an open set* G *such that* $C(G) < \epsilon$ *and* $f|_{G^c} \in C(G^c)$.

For $0 < p_0 < \infty$ consider the spaces $L^{p_0}(C)$ and $L^\infty(C)$. For every $t > 0$ we have that

$$\begin{aligned} K(t, f; L^{p_0}(C), L^\infty(C)) &\leq K(t, f; L^{p_0}(C) \cap QC, L^\infty(C) \cap QC) \\ &=: K_{QC}(t, f; L^{p_0}(C), L^\infty(C)). \end{aligned}$$

Proposition 1. *If f is nonnegative, then*

$$K(t,f;L^{p_0}(C),L^\infty(C)) = \inf_{\lambda>0}\left(\|(f-\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\lambda)\|_{L^\infty(C)}\right).$$

Proof. By definition, since $f=(f-\lambda)_+ + \min(f,\lambda)$ for all $\lambda>0$, we have that

$$K(t,f;L^{p_0}(C),L^\infty(C)) \le \inf_{\lambda>0}\left(\|(f-\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\lambda)\|_{L^\infty(C)}\right).$$

To prove the reversed estimate, let $\epsilon>0$ and choose $f_0, f_1 \ge 0$ such that $f=f_0+f_1$ and

$$\|f_0\|_{L^{p_0}(C)} + t\|f_1\|_{L^\infty(C)} \le K(t,f;L^{p_0}(C),L^\infty(C)) + \epsilon.$$

If $\overline{\lambda} = \|f_1\|_{L^\infty(C)}$, then $f-\overline{\lambda} \le f_0$ and $0 \le (f-\overline{\lambda})_+ \le f_0$. Hence $\|(f-\overline{\lambda})_+\|_{L^{p_0}(C)} \le \|f_0\|_{L^{p_0}(C)}$, and also $\|\min(f,\overline{\lambda})\|_{L^\infty(C)} \le \|f_1\|_{L^\infty(C)}$. Thus

$$\begin{aligned}
&\inf_{\lambda>0}\left(\|(f-\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\lambda)\|_{L^\infty(C)}\right)\\
&\le \|(f-\overline{\lambda})_+\|_{L^{p_0}(C)} + t\|\min(f,\overline{\lambda})\|_{L^\infty(C)}\\
&\le \|f_0\|_{L^{p_0}(C)} + t\|f_1\|_{L^\infty(C)}\\
&\le K(t,f;L^{p_0}(C),L^\infty(C)) + \epsilon
\end{aligned}$$

and the estimate follows. □

Proposition 2. *If $f \in QC$ is nonnegative, then*

$$K_{QC}(t,f;L^{p_0}(C),L^\infty(C)) = K(t,f;L^{p_0}(C),L^\infty(C)).$$

Proof. If $f \in QC$ is nonnegative, then for all $\lambda>0$ we have that $(f-\lambda)_+ \in QC$ and $\min(f,\lambda) \in QC$ since they are nonnegative. Then, for all $\lambda>0$,

$$K_{QC}(t,f;L^{p_0}(C),L^\infty(C)) \le \|(f-\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\lambda)\|_{L^\infty(C)}$$

and hence

$$\begin{aligned}
&K_{QC}(t,f;L^{p_0}(C),L^\infty(C))\\
&\le \inf_{\lambda>0}\left(\|(f-\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\lambda)\|_{L^\infty(C)}\right)\\
&= K(t,f;L^{p_0}(C),L^\infty(C))
\end{aligned}$$

by the Proposition 1. □

Let us now show that $K(t,f;L^{p_0}(C),L^\infty(C)) = K(t,|f|;L^{p_0}(C),L^\infty(C))$. Obviously, $K(t,f;L^{p_0}(C),L^\infty(C)) \le K(t,|f|;L^{p_0}(C),L^\infty(C))$. To prove the reversed inequality, let $\epsilon>0$ and choose $f_0 \in L^{p_0}(C)$, $f_1 \in L^\infty(C)$ such that $f=f_0+f_1$ and

$$\|f_0\|_{L^{p_0}(C)} + t\|f_1\|_{L^\infty(C)} \le K(t,f;L^{p_0}(C),L^\infty(C)) + \epsilon.$$

Define

$$s(f) := \begin{cases} 1, \text{if } f(x) \ge 0 \\ -1, \text{if } f(x) < 0. \end{cases}$$

Then $s(f)f = s(f)(f_0+f_1) = s(f)f_0 + s(f)f_1$, which means that $|f| = s(f)f_0 + s(f)f_1$, being $\|f_0\|_{L^{p_0}(C)} = \|s(f)f_0\|_{L^{p_0}(C)}$ and $\|f_1\|_{L^\infty(C)} =$

$\|s(f)f_1\|_{L^\infty(C)}$. Then

$$\begin{aligned} K(t,|f|;L^{p_0}(C),L^\infty(C)) &\le \|s(f)f_0\|_{L^{p_0}(C)} + t\|s(f)f_1\|_{L^\infty(C)} \\ &\le K(t,f;L^{p_0}(C),L^\infty(C)) + \epsilon \end{aligned}$$

and we get that

$$K(t,|f|;L^{p_0}(C),L^\infty(C)) \le K(t,f;L^{p_0}(C),L^\infty(C)).$$

Since $K(t,|f|;L^{p_0}(C),L^\infty(C)) = K(t,f;L^{p_0}(C),L^\infty(C))$, we conclude that

$$\||f|\|_{(L^{p_0}(C),L^\infty(C))_{\theta,q}} = \|f\|_{(L^{p_0}(C),L^\infty(C))_{\theta,q}}.$$

PROPOSITION 3. *Let f be a quasicontinuous function, not necessarily positive, then*

$$K_{QC}(t,f;L^{p_0}(C),L^\infty(C)) \simeq K(t,f;L^{p_0}(C),L^\infty(C)).$$

Proof.

$$\begin{aligned} K_{QC}(t,f;L^{p_0}(C),L^\infty(C)) &= K_{QC}(t,f^+-f^-;L^{p_0}(C),L^\infty(C)) \\ &\le K_{QC}(t,f^+;L^{p_0}(C),L^\infty(C)) + K_{QC}(t,f^-;L^{p_0}(C),L^\infty(C)) \\ &\le K_{QC}(t,|f|;L^{p_0}(C),L^\infty(C)) + K_{QC}(t,|f|;L^{p_0}(C),L^\infty(C)) \\ &= 2K_{QC}(t,|f|;L^{p_0}(C),L^\infty(C)) = 2K(t,|f|;L^{p_0}(C),L^\infty(C)) \\ &= 2K(t,f;L^{p_0}(C),L^\infty(C)) \le 2K_{QC}(t,f;L^{p_0}(C),L^\infty(C)) \end{aligned}$$

since $|f| \in QC$. $\square$

Thus, for $0<\theta<1$ and $q>0$, we have that

$$\begin{aligned} \|f\|_{(L^{p_0}(C)\cap QC,L^\infty(C)\cap QC)_{\theta,q}} &:= \left(\int_0^\infty (t^{-\theta}K_{QC}(t,f))^q \frac{dt}{t}\right)^{1/q} \\ &\lesssim \||f|\|_{(L^{p_0}(C),L^\infty(C))_{\theta,q}}. \end{aligned}$$

Hence

$$(L^{p_0}(C),L^\infty(C))_{\theta,q}\cap QC \hookrightarrow (L^{p_0}(C)\cap QC, L^\infty(C)\cap QC)_{\theta,q}$$

and therefore

$$(L^{p_0}(C),L^\infty(C))_{\theta,q}\cap QC = (L^{p_0}(C)\cap QC, L^\infty(C)\cap QC)_{\theta,q}. \tag{1}$$

By denoting $\mathfrak{L}^{p,q}(C) = L^{p,q}(C)\cap QC$, we obtain the following result:

THEOREM 2. *Supposse that $0<\theta<1$, $0<p_0<q\le\infty$ or $0<p_0\le q<\infty$ and $\frac{1}{p} := \frac{1-\theta}{p_0}$, then*

$$(\mathfrak{L}^{p_0}(C),\mathfrak{L}^\infty(C))_{\theta,q} = \mathfrak{L}^{p,q}(C).$$

Proof.

$$\begin{aligned} (\mathfrak{L}^{p_0}(C),\mathfrak{L}^\infty(C))_{\theta,q} &= (L^{p_0}(C)\cap QC, L^\infty(C)\cap QC)_{\theta,q} \\ &= (L^{p_0}(C),L^\infty(C))_{\theta,q}\cap QC = L^{p,q}(C)\cap QC := \mathfrak{L}^{p,q}(C) \end{aligned}$$

by (1) and Theorem 1. □

Corollary 2. *Take* $0 < p_0, p_1, q_0, q_1 < \infty$, $p_0 \neq p_1$ *and* $0 < \eta < 1$. *Then*

$$(\mathfrak{L}^{p_0,q_0}(C), \mathfrak{L}^{p_1,q_1}(C))_{\eta,q} = \mathfrak{L}^{p,q}(C)$$

with $\frac{1}{p} := \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$.

Proof. Let $0 < r < \min(p_0, p_1, q_0, q_1)$ and choose $1 - \theta_i := \frac{r}{p_i}$, $i = 0, 1$. Then, if $\theta = (1-\eta)\theta_0 + \eta\theta_1$, since $\frac{1}{p} = \frac{1-\theta}{r}$ we get

$$\begin{aligned}(\mathfrak{L}^{p_0,q_0}(C), \mathfrak{L}^{p_1,q_1}(C))_{\eta,q} &= ((\mathfrak{L}^r(C), \mathfrak{L}^\infty(C))_{\theta_0,q_0}, (\mathfrak{L}^r(C), \mathfrak{L}^\infty(C))_{\theta_1,q_1})_{\eta,q} \\ = (\mathfrak{L}^r(C), \mathfrak{L}^\infty(C))_{\theta,q} &= \mathfrak{L}^{p,q}(C) = L^{p,q}(C) \cap QC \\ &= (L^{p_0,q_0}(C), L^{p_1,q_1}(C))_{\eta,q} \cap QC\end{aligned}$$

by Theorem 2, Theorem 3.11.5 of [3], and Corollary 1. □

Acknowledgments. Partially supported by DGICYT grant MTM2007-60500.

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