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ON THE RATE OF SUMMABILITY BY MATRIX MEANS IN THE GENERALIZED HÖLDER METRIC

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Abstract. We will generalize and improve the results of T. Singh [Degree of approximation to functions in a normed spaces, Publ. Math. Debrecen, 40/3-4, (1992), 261-271] obtaining the L. Leindler type estimates from [On the degree of approximation of continuous functions, Acta Math. Hungar., 104 (1-2), (2004), 105-113].

1. Introduction. Let f be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

be its Fourier series. Denote by $S_n(x) = S_n(f, x)$ the n-th partial sum of (1.1) and by $\omega(f, \delta)$ the modulus of continuity of $f \in C_{2\pi}$.

The usual supremum norm will be denoted by $\|\cdot\|_C$.

Let ω be a nondecreasing continuous function on the interval $[0, 2\pi]$ having the properties

$$\omega(0) = 0, \; \omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2).$$

Such function will be called modulus of continuity.

Denote by H^ω the class of functions

$$H^\omega := \{f \in C_{2\pi}; \; |f(x) - f(y)| \le C\omega(|x-y|)\},$$

where C is a positive constant. For $f \in H^\omega$, we define the norm $\|\cdot\|_\omega$ by the formula

$$\|f\|_\omega := \|f\|_C + \sup_{x,\, y} |\Delta^\omega f(x, y)|,$$

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where

$$\Delta^{\omega} f(x,y) = \frac{|f(x) - f(y)|}{\omega(|x-y|)}, x \neq y$$

and $\Delta^{0} f(x,y) = 0$. If $\omega(t) = C_1 |t|^{\alpha}$ $(0 < \alpha \leq 1)$, where C_1 is a positive constants, then

$$H^{\alpha} = \{f \in C_{2\pi};\ \ |f(x) - f(y)| \leq C_1 |x-y|^{\alpha},\ 0 < \alpha \leq 1\}$$

is a Banach space and the metric induced by the norm $\|\cdot\|_{\alpha}$ on H^{α} is said to be a Hölder metric.

Let $A := (a_{nk})$ $(k, n = 0, 1, ...)$ be a lower triangular infinite matrix of real numbers satisfying the following condition:

$$a_{nk} \geq 0\ \ (k, n = 0, 1, ...), a_{nk} = 0, k > n \text{ and } \sum_{k=0}^{n} a_{nk} = 1, \tag{1.2}$$

Let the $A-$transformation of $(S_n(f;x))$ be given by

$$T_n(f) := T_n(f;x) := \sum_{k=0}^{n} a_{nk} S_k(f;x)\ \ (n = 0, 1, ...). \tag{1.3}$$

Now, we define two classes of sequences (see [3]).

A sequence $c := (c_n)$ of nonnegative numbers tending to zero is called the Rest Bounded Variation Sequence, or briefly $c \in RBVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_n - c_{n+1}| \leq K(c)\, c_m \tag{1.4}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $c := (c_n)$ of nonnegative numbers will be called the Head Bounded Variation Sequence, or briefly $c \in HBVS$, if it has the property

$$\sum_{k=0}^{m-1} |c_n - c_{n+1}| \leq K(c)\, c_m \tag{1.5}$$

for all natural numbers m, or only for all $m \leq N$ if the sequence c has only finite number ofnonzero terms and the last nonzero term is c_N.

Therefore we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all n, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (1.4) or (1.5) for the sequence $\alpha_n := (a_{nk})_{k=0}^{\infty}$. Now, we can give the conditions to be used later on. We assume that for all n and $0 \leq m \leq n$

$$\sum_{k=m}^{\infty} |a_{nk} - a_{nk+1}| \leq K a_{nm} \tag{1.6}$$

and

$$\sum_{k=0}^{m-1} |a_{nk} - a_{nk+1}| \leq K a_{nm} \tag{1.7}$$

hold if $\alpha_n := (a_{nk})_{k=0}^{\infty}$ belongs to $RBVS$ or $HBVS$, respectively.

Let ω and ω^* be two given moduli of continuity satisfying the following condition (for $0 \le p < q \le 1$):

$$\frac{(\omega(t))^{\frac{p}{q}}}{\omega^*(t)} = O(1) \quad (t \to 0_+). \tag{1.8}$$

In [4] R. Mohapatra and P. Chandra obtained some results on degree of approximation by the means (1.3) in the Hölder metric. Recently, T. Singh in [5] established the following two theorems generalizing of some results of P. Chandra [1] with a mediate function H such that:

$$\int_u^{\pi} \frac{\omega(f;t)}{t^2} dt = O(H(u)) \quad (u \to 0_+), \; H(t) \ge 0 \tag{1.9}$$

and

$$\int_0^t H(u)\,du = O(tH(t)) \quad (t \to O_+). \tag{1.10}$$

THEOREM 1.1 *[5] Let $A = (a_{nk})$ satisfy the condition (1.2) and $a_{nk} \le a_{nk+1}$ $(k = 0, 1, ..., n-1;\; n = 0, 1, ...)$. Then for $f \in H^{\omega}$, $0 \le p < q \le 1$*

$$\|T_n(f) - f\|_{\omega^*} = O\left[\{\omega(|x-y|)\}^{\frac{p}{q}} \{\omega^*(|x-y|)\}^{-1} \right.$$
$$\left. \times \left\{ \left(H\left(\frac{\pi}{n}\right)\right)^{1-\frac{p}{q}} a_{nn} \left(n^{\frac{p}{q}} + a_{nn}^{-\frac{p}{q}} \right) \right\} \right] + O\left(a_{nn} H\left(\frac{\pi}{n}\right)\right), \tag{1.11}$$

if $\omega(t)$ satisfies (1.9) and (1.10), and

$$\|T_n(f) - f\|_{\omega^*} = O\left[\{\omega(|x-y|)\}^{\frac{p}{q}} \{\omega^*(|x-y|)\}^{-1}\right]$$
$$\times \left\{ \left(\omega\left(\frac{\pi}{n}\right)\right)^{1-\frac{p}{q}} + a_{nn} n^{\frac{p}{q}} \left(H\left(\frac{\pi}{n}\right)\right)^{1-\frac{p}{q}} \right\} + O\left\{\omega\left(\frac{\pi}{n}\right) + a_{nn} H\left(\frac{\pi}{n}\right)\right\}, \tag{1.12}$$

if $\omega(t)$ satisfies (1.9).

THEOREM 1.2 *[5] Let $A = (a_{nk})$ satisfy the condition (1.2) and $a_{nk} \le a_{nk+1}$ $(k = 0, 1, ..., n-1;\; n = 0, 1, ...)$ and let $\omega(f;t)$ satisfies (1.9) and (1.10). Then for $f \in H^{\omega}$, $0 \le p < q \le 1$*

$$\|T_n(f) - f\|_{\omega^*} = O\left[\{\omega(|x-y|)\}^{\frac{p}{q}} \{\omega^*(|x-y|)\}^{-1} \right.$$
$$\left. \times \left\{ (H(a_{n0}))^{1-\frac{p}{q}} a_{n0} \left(n^{\frac{p}{q}} + a_{n0}^{-\frac{p}{q}} \right) \right\} \right] + O(a_{n0} H(a_{n0})). \tag{1.13}$$

Another generalization of the results of Chandra [2] was obtained by L. Leindler in [3]. Namely, he proved following theorems.

THEOREM 1.3 *[3] Let (1.2), (1.7) and (1.9) hold. Then for $f \in C_{2\pi}$*

$$\|T_n(f) - f\|_C = O\left(\omega\left(\frac{\pi}{n}\right)\right) + O\left(a_{nn} H\left(\frac{\pi}{n}\right)\right). \tag{1.14}$$

If, in addition, $\omega(f;t)$ satisfies the condition (1.10) then

$$\|T_n(f) - f\|_C = O(a_{nn} H(a_{nn})). \tag{1.15}$$

THEOREM 1.4 *[3] Let (1.2), (1.6), (1.9) and (1.10) hold. Then for* $f \in C_{2\pi}$

$$\|T_n(f) - f\|_C = O\left(a_{n0} H\left(a_{n0}\right)\right). \tag{1.16}$$

In the presented paper we will generalize and improve the results of T. Singh [5] obtaining the L. Leindler type estimates from [3] in the generalized Hölder metric instead of the supremum norm.

Throughout the paper we shall use the following notations:

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x),$$

$$A_{nk} = \sum_{r=n-k+1}^{n} a_{nr}, \quad (k = 1, 2, ..., n+1),$$

$$A_n(u) = \sum_{k=0}^{n} a_{nk} \frac{\sin\left(k + \frac{1}{2}\right) u}{\sin \frac{1}{2} u}.$$

By K_1, $K_2, \ldots$ we shall designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same at each occurrence.

2. Main results. Our main results are the following.

THEOREM 2.1 Let (1.2), (1.7) and (1.8) hold. Suppose $\omega(f;t)$ satisfies (1.9), then for $f \in H^{\omega}$

$$\|T_n(f) - f\|_{\omega^*} = O\left(\left\{\sum_{k=1}^{n+1} \frac{A_{nk}}{k}\right\}^{\frac{p}{q}} \left\{a_{nn} H\left(\frac{\pi}{n}\right)\right\}^{1-\frac{p}{q}}\right). \tag{2.1}$$

If, in addition, $\omega(f;t)$ satisfies the condition (1.10), then

$$\|T_n(f) - f\|_{\omega^*} = O\left(\left\{\sum_{k=1}^{n+1} \frac{A_{nk}}{k}\right\}^{\frac{p}{q}} \left\{a_{nn} H\left(a_{nn}\right)\right\}^{1-\frac{p}{q}}\right). \tag{2.2}$$

THEOREM 2.2 *Let (1.2), (1.8), (1.6) and (1.9) hold. Then, for* $f \in H^{\omega}$

$$\|T_n(f) - f\|_{\omega^*} = O\left(\left\{a_{n0} H\left(\frac{\pi}{n}\right)\right\}^{1-\frac{p}{q}}\right). \tag{2.3}$$

If, in addition, $\omega(f;t)$ *satisfies (1.10), then*

$$\|T_n(f) - f\|_{\omega^*} = O\left(\left\{a_{n0} H\left(a_{n0}\right)\right\}^{1-\frac{p}{q}}\right). \tag{2.4}$$

REMARK 2.1 We can observe, that under the condition (1.8), Theorems 1.1 and 1.2 are the corollaries of Theorems 2.1 and 2.2, respectively. The assumption $a_{nk} \leq a_{nk+1}$ $(k = 0, 1, ..., n-1;\ n = 0, 1, ...)$ of Theorem 1.1 implies the inequality

$$\sum_{k=1}^{n+1} \frac{A_{nk}}{k} \leq (n+1)\, a_{nn},$$

whence by the Theorem 2.1, we obtain the relation of the (1.11) type. The estimate (1.13) from Theorem 1.2 are also the consequences of the estimate of Theorem 2.2 and sometimes are better since $(n a_{nn})$ can be bounded.

REMARK 2.2 If in the assumptions of the Theorem 2.1 or 2.2 we take $\omega(|t|) = O(|t|^q)$, $\omega^*(|t|) = O(|t|^p)$ with $p = 0$, then from (2.1), (2.2) and (2.3), we have the estimations (1.14), (1.15) and (1.16), respectively.

3. Lemmas. To prove our theorems we need the following lemmas.

LEMMA 3.1 [2] If (1.9) and (1.10) hold then

$$\int_0^r \frac{\omega(f;t)}{t} dt = O(rH(r)) \quad (r \to 0_+). \tag{3.1}$$

LEMMA 3.2 [7] If (1.7) holds, then for $\frac{1}{n} \le u \le \pi$

$$|A_n(u)| \le \frac{\pi^2 (K+1)^2 + \pi}{u} A_{n,u^{-1}}, \tag{3.2}$$

where $u^{-1} := \max\{1, [u^{-1}]\}$.

LEMMA 3.3 [7] If (1.6) holds, then for $f \in C_{2\pi}$

$$\|T_n(f) - f\|_C \le 8(K+1)(2K+1) \sum_{k=0}^{n} a_{nk} E_k(f), \tag{3.3}$$

where $E_n(f)$ denotes the best approximation of function f by trigonometric polynomials of order at most n.

LEMMA 3.4 [7] If (1.6) holds, then

$$\int_0^\pi |A_n(t)| dt \le 4K(K+1). \tag{3.4}$$

LEMMA 3.5 If (1.2), (1.6) hold and $\omega(f;t)$ satisfies (1.9) then

$$\sum_{k=0}^{n} a_{nk} \omega\left(f; \frac{\pi}{k+1}\right) = O\left(a_{n0} H\left(\frac{\pi}{n}\right)\right). \tag{3.5}$$

If, in addition, $\omega(f;t)$ satisfies (1.10) then

$$\sum_{k=0}^{n} a_{nk} \omega\left(f; \frac{\pi}{k+1}\right) = O(a_{n0} H(a_{n0})). \tag{3.6}$$

Proof. First we prove (3.5). If (1.6) holds, then

$$a_{nn} - a_{nm} \le |a_{nm} - a_{nn}| \le \sum_{k=m}^{n-1} |a_{nk} - a_{nk+1}|$$
$$\le \sum_{k=m}^{\infty} |a_{nk} - a_{nk+1}| \le K a_{nm}$$

for any $n \ge m \ge 0$, whence

$$a_{nn} \le (K+1) a_{nm}. \tag{3.7}$$

From this, using (1.9), we get

$$\sum_{k=0}^{n} a_{nk}\omega\left(f;\frac{\pi}{k+1}\right) \leq (K+1)\,a_{n0}\sum_{k=0}^{n}\omega\left(f;\frac{\pi}{k+1}\right)$$

$$\leq K_1 a_{n0}\int_{1}^{n+1}\omega\left(f;\frac{\pi}{t}\right)dt = K_1 a_{n0}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\omega(f;u)}{u^2}du = O\left(a_{n0}H\left(\frac{\pi}{n}\right)\right).$$

Now, we prove (3.6). Since

$$(K+1)(n+1)\,a_{n0} \geq \sum_{k=0}^{n} a_{nk} = 1$$

we can see that

$$\sum_{k=0}^{n} a_{nk}\omega\left(f;\frac{\pi}{k+1}\right) \leq \sum_{k=0}^{\frac{1}{(K+1)a_{n0}}-1} a_{nk}\omega\left(f;\frac{\pi}{k+1}\right)$$

$$+\sum_{k=\frac{1}{(K+1)a_{n0}}-1}^{n} a_{nk}\omega\left(f;\frac{\pi}{k+1}\right).$$

Using again (3.7), (1.2) and the monotonicity of the modulus of continuity, we obtain

$$\sum_{k=0}^{n} a_{nk}\omega\left(f;\frac{\pi}{k+1}\right) \leq (K+1)\,a_{n0}\sum_{k=0}^{\frac{1}{(K+1)a_{n0}}-1}\omega\left(f;\frac{\pi}{k+1}\right)$$

$$+\omega\left(f;\pi\,(K+1)\,a_{no}\right)\sum_{k=\frac{1}{(K+1)a_{n0}}-1}^{n} a_{nk}$$

$$\leq K_1 a_{n0}\int_{1}^{\frac{1}{(K+1)a_{n0}}}\omega\left(f;\frac{\pi}{t}\right)dt + \omega\left(f;\pi\,(K+1)\,a_{no}\right)$$

$$\leq K_1 a_{n0}\int_{a_{n0}}^{\pi}\frac{\omega(f;u)}{u^2}du + 2\pi\,(K+1)\,\omega\left(f;a_{n0}\right). \tag{3.8}$$

According to

$$\omega\left(f;a_{n0}\right) \leq 4\omega\left(f;\frac{a_{n0}}{2}\right) \leq 8\int_{\frac{a_{n0}}{2}}^{a_{n0}}\frac{\omega(f;t)}{t}dt \leq 8\int_{0}^{a_{n0}}\frac{\omega(f;t)}{t}dt,$$

in view of (3.8), (1.9) and (1.10), the relation (3.6) holds. ∎

4. Proofs of the Theorems. In this section we shall prove the Theorems 2.1 and 2.2.

4.1. Proof of Theorem 2.1. First we prove (2.1). Setting

$$R_n(x) = T_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi_x(t)\, A_n(t)\, dt$$

and

$$R_n(x,y) = R_n(x) - R_n(y) = \frac{1}{2\pi}\int_0^\pi (\phi_x(t) - \phi_y(t))\, A_n(t)\, dt$$

we get

$$|R_n(x,y)| \le \frac{1}{2\pi}\int_0^\pi |\phi_x(t) - \phi_y(t)|\, |A_n(t)|\, dt.$$

It is clear that

$$|\phi_x(t) - \phi_y(t)| \le 4C\omega(|x-y|) \tag{4.1}$$

and

$$|\phi_x(t) - \phi_y(t)| \le 4\omega(f;|t|). \tag{4.2}$$

Then, using (4.1), we have

$$|R_n(x,y)| \le \frac{2C}{\pi}\omega(|x-y|)\left(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi}\right)|A_n(t)|\, dt = \frac{2C}{\pi}\omega(|x-y|)(I_1 + I_2). \tag{4.3}$$

It is obvious that

$$\begin{aligned} I_1 &\le \int_0^{\frac{\pi}{n}} \frac{1}{\sin\frac{1}{2}t}\sum_{k=0}^{n} a_{nk}\left|\sin\left(k+\frac{1}{2}\right)t\right| dt \\ &\le \pi\int_0^{\frac{\pi}{n}}\sum_{k=0}^{n} a_{nk}\left(k+\frac{1}{2}\right) dt \le \frac{3}{2}\pi^2. \end{aligned} \tag{4.4}$$

Using (3.2), we obtain

$$\begin{aligned} I_2 &\le K_1\int_{\frac{\pi}{n}}^{\pi}\frac{A_{n,t^{-1}}}{t}dt = K_1\int_{\frac{1}{\pi}}^{\frac{n}{\pi}}\frac{A_{n,t}}{t}dt \\ &= K_1\sum_{k=1}^{n-1}\int_{\frac{k}{\pi}}^{\frac{k+1}{\pi}}\frac{A_{n,t}}{t}dt \le K_1\sum_{k=1}^{n-1}\frac{A_{n,k+1}}{k} \\ &\le 2K_1\sum_{k=2}^{n}\frac{A_{n,k}}{k} \le 2K_1\sum_{k=1}^{n+1}\frac{A_{n,k}}{k}. \end{aligned} \tag{4.5}$$

If (1.7), holds then

$$a_{n\mu} - a_{nm} \le |a_{n\mu} - a_{nm}| \le \sum_{k=\mu}^{m-1}|a_{nk} - a_{nk+1}| \le Ka$$

for any $m \geq \mu \geq 0$, whence

$$a_{n\mu} \leq (K+1)\, a_{nm}. \tag{4.6}$$

From this and (1.2) we can observe that

$$\sum_{k=1}^{n+1} \frac{A_{n,k}}{k} = \sum_{k=1}^{n+1} \frac{1}{k} \sum_{r=n-k+1}^{n} a_{nr}$$

$$\geq \frac{1}{K+1} \sum_{k=1}^{n+1} a_{n,n-k+1} = \frac{1}{K+1} \sum_{k=0}^{n} a_{nk} = \frac{1}{K+1}$$

and by (4.3)-(4.5), we obtain

$$|R_n(x,y)| \leq K_2 \omega(|x-y|) \sum_{k=1}^{n+1} \frac{A_{n,k}}{k}. \tag{4.7}$$

On the other hand, by (4.2), we have

$$|R_n(x,y)| \leq \frac{2}{\pi} \int_0^{\pi} \omega(f;t)\, |A_n(t)|\, dt$$

$$= \frac{2}{\pi} \left(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi} \right) \omega(f;t)\, |A_n(t)|\, dt = \frac{2}{\pi} \left(I_1' + I_2' \right). \tag{4.8}$$

Using (4.6) and (1.9), we can estimate the quantities I_1' and I_2' as follow:

$$I_1' \leq \omega\left(f; \frac{\pi}{n}\right) \int_0^{\frac{\pi}{n}} \frac{1}{\sin \frac{1}{2} t} \sum_{k=0}^{n} a_{nk} \left| \sin\left(k + \frac{1}{2}\right) t \right| dt$$

$$\leq \frac{3}{2} \pi^2 \omega\left(f; \frac{\pi}{n}\right) \sum_{k=0}^{n} a_{nk} \leq 3\pi^2 (K+1) a_{nn} \sum_{k=1}^{n} \omega\left(f; \frac{\pi}{k}\right)$$

$$\leq K_3 a_{nn} \int_1^n \omega\left(f; \frac{\pi}{t}\right) dt = K_3 a_{nn} \int_{\frac{\pi}{n}}^{\pi} \frac{\omega(f;u)}{u^2} du = O\left(a_{nn} H\left(\frac{\pi}{n}\right)\right) \tag{4.9}$$

and, by (3.2),

$$I_2' \leq K_4 \int_{\frac{\pi}{n}}^{\pi} \omega(f;t) \frac{A_{n,t^{-1}}}{t} dt \leq K_5 a_{nn} \int_{\frac{\pi}{n}}^{\pi} \frac{\omega(f;t)}{t^2} dt = O\left(a_{nn} H\left(\frac{\pi}{n}\right)\right). \tag{4.10}$$

Combining (4.8)-(4.10) we obtain

$$|R_n(x,y)| = O\left(a_{nn} H\left(\frac{\pi}{n}\right)\right). \tag{4.11}$$

Therefore, using (4.7) and (4.11),

$$\sup_{x,\ y} \left\{ \Delta^{\omega^*} R_n(x,y) \right\} = \sup_{x,\ y} \left\{ \frac{|R_n(x,y)|^{\frac{p}{q}}}{\omega^*(|x-y|)} |R_n(x,y)|^{1-\frac{p}{q}} \right\}$$

$$\leq K_4 \left\{ \sum_{k=1}^{n+1} \frac{A_{nk}}{k} \right\}^{\frac{p}{q}} \left\{ a_{nn} H\left(\frac{\pi}{n}\right) \right\}^{1-\frac{p}{q}}. \tag{4.12}$$

Since

$$|R_n(x)| \leq \frac{1}{2\pi} \int_0^\pi |\phi_x(t)| |A_n(t)| \, dt \leq \frac{1}{\pi} \int_0^\pi \omega(f;t) |A_n(t)| \, dt,$$

the inequalities (4.4), (4.5), (4.8) and (4.9) lead us to

$$\|T_n(f) - f\|_C \leq \frac{1}{\pi} \left\{ \int_0^\pi \omega(f;t) |A_n(t)| \, dt \right\}^{\frac{p}{q}} \left\{ \int_0^\pi \omega(f;t) |A_n(t)| \, dt \right\}^{1-\frac{p}{q}}$$

$$\leq \frac{1}{\pi} (\omega(f;\pi))^{\frac{p}{q}} \left\{ \int_0^\pi |A_n(t)| \, dt \right\}^{\frac{p}{q}} \left\{ \int_0^\pi \omega(f;t) |A_n(t)| \, dt \right\}^{1-\frac{p}{q}}$$

$$= \frac{1}{\pi} (\omega(f;\pi))^{\frac{p}{q}} \left\{ \left(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^\pi \right) |A_n(t)| \, dt \right\}^{\frac{p}{q}} \left\{ \left(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^\pi \right) \omega(f;t) |A_n(t)| \, dt \right\}^{1-\frac{p}{q}}$$

$$\leq K_5 \left\{ \sum_{k=1}^{n+1} \frac{A_{nk}}{k} \right\}^{\frac{p}{q}} \left\{ a_{nn} H\left(\frac{\pi}{n}\right) \right\}^{1-\frac{p}{q}}. \tag{4.13}$$

Collecting our partial results (4.12) and (4.13), we obtain that (2.1) holds.

Now, we prove (2.2). By (4.2) we have

$$|R_n(x,y)| \leq \frac{2}{\pi} \int_0^\pi \omega(f;t) |A_n(t)| \, dt$$

$$= \frac{2}{\pi} \left(\int_0^{a_{nn}} + \int_{a_{nn}}^\pi \right) \omega(f;t) |A_n(t)| \, dt = \frac{2}{\pi} (J_1 + J_2). \tag{4.14}$$

Using (1.9) and (1.10), we shall estimate the quantities J_1 and J_2 similarly like the quantities I_1' and I_2', respectively. Namely, by Lemma 3.1,

$$J_1 \leq \pi \int_0^{a_{nn}} \frac{\omega(f;t)}{t} dt = O(a_{nn} H(a_{nn}))$$

and, by (3.2),

$$J_2 \leq K_6 \int_{a_{nn}}^\pi \omega(f;t) \frac{A_{n,t^{-1}}}{t} dt \leq K_7 a_{nn} \int_{a_{nn}}^\pi \frac{\omega(f;t)}{t^2} dt = O(a_{nn} H(a_{nn})).$$

From this and (4.12) we get

$$\sup_{x,\ y} \left\{ \Delta^{\omega^*} R_n(x,y) \right\} = O\left(\left\{ \sum_{k=1}^{n+1} \frac{A_{nk}}{k} \right\}^{\frac{p}{q}} \{a_{nn} H(a_{nn})\}^{1-\frac{p}{q}} \right). \tag{4.15}$$

In the same manner as in (4.13) we can show that

$$\|T_n(f) - f\|_C \le K_5 \left\{ \sum_{k=1}^{n+1} \frac{A_{nk}}{k} \right\}^{\frac{p}{q}} \{a_{nn} H(a_{nn})\}^{1-\frac{p}{q}}. \tag{4.16}$$

Combining (4.15) and (4.16) we conclude that (2.2) holds. This completes the proof. ∎

4.2. Proof of Theorem 2.2. Using the same notations as in the proof of Theorem 2.1, from (4.1) and (3.4), we get

$$\begin{aligned} |R_n(x,y)| &\le \frac{2C}{\pi} \omega(|x-y|) \int_0^{\pi} |A_n(t)|\, dt \\ &\le \frac{8CK(K+1)}{\pi} \omega(|x-y|). \end{aligned} \tag{4.17}$$

On the other hand

$$|R_n(x,y)| \le |T_n(f;x) - f(x)| + |T_n(f;y) - f(y)|.$$

The estimate (3.3) and the inequality

$$E_n(f) \le K_1 \omega\left(f; \frac{\pi}{n+1}\right),$$

give

$$\begin{aligned} |R_n(x,y)| &\le 16(K+1)(2K+1) \sum_{k=0}^{n} a_{nk} E_k(f) \\ &\le K_2 \sum_{k=0}^{n} a_{nk} \omega\left(f, \frac{\pi}{k+1}\right). \end{aligned} \tag{4.18}$$

Therefore, by (4.17),

$$\begin{aligned} \sup_{x,\ y} \left\{ \Delta^{\omega^*} R_n(x,y) \right\} &= \sup_{x,\ y} \left\{ \frac{|R_n(x,y)|^{\frac{p}{q}}}{\omega^*(|x-y|)} |R_n(x,y)|^{1-\frac{p}{q}} \right\} \\ &\le K_3 \left\{ \sum_{k=0}^{n} a_{nk} \omega\left(f, \frac{\pi}{k+1}\right) \right\}^{1-\frac{p}{q}}. \end{aligned} \tag{4.19}$$

The same estimate can be shown for the deviation $T_n(f;x) - f(x)$. Namely, by (3.3) and (1.2), we get

$$\begin{aligned} \|T_n(f) - f\|_C &\le 8(K+1)(2K+1) \sum_{k=0}^{n} a_{nk} E_k(f) \le K_4 \sum_{k=0}^{n} a_{nk} \omega\left(f, \frac{\pi}{k+1}\right) \\ &\le K_5 \left\{ \sum_{k=0}^{n} a_{nk} \omega\left(f, \frac{\pi}{k+1}\right) \right\}^{\frac{p}{q}} \left\{ \sum_{k=0}^{n} a_{nk} \omega\left(f, \frac{\pi}{k+1}\right) \right\}^{1-\frac{p}{q}} \\ &\le K_6 \left\{ \sum_{k=0}^{n} a_{nk} \omega\left(f, \frac{\pi}{k+1}\right) \right\}^{1-\frac{p}{q}}. \end{aligned} \tag{4.20}$$

Finally, collecting our partial results (4.19) and (4.20) and using Lemma 3.5 we obtain that (2.3) and (2.4) hold. ■

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