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A STRENGTHENING OF A THEOREM OF MARCINKIEWICZ

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Abstract. We consider a problem of intervals raised by I.Ya. Novikov in [2], which refines the well-known theorem of J. Marcinkiewicz concerning structure of closed sets [3, Chapt. IV, Theorem 2.1]. A positive solution to the problem for some specific cases is obtained. As a result, we strengthen the theorem of Marcinkiewicz for generalized Cantor sets.

1. Introduction. Let us start with some basic notations. For any set $M \subset \mathbb{R}$ by μM we will denote the usual (Lebesgue) measure of M. Further, by $\sup M$ ($\inf M$) we will mean the least upper bound (the biggest lower bound) of M, respectively. We will write $\max M$, $\min M$ instead of $\sup M$, $\inf M$ when $\sup M \in M$, $\inf M \in M$.

Let F be a closed bounded nowhere dense set on $\mathbb{R}$ with a positive measure and let $\delta(y)$ denote a distance from y to F. The integral of Marcinkiewicz is defined as follows:

$$I_\lambda(x) = \int\limits_{\min F}^{\max F} \frac{\delta(y)^\lambda}{|x-y|^{1+\lambda}} dy. \tag{1}$$

As is well-known, for each $\lambda > 0$ the integral (1) converges for almost all points of F [1], [3, Chapt. IV, Theorem 2.1]. The result can be reformulated as follows (see [3, p. 131]): let $\{(a_i; a_i + \delta_i)\}_{i=1}^\infty$ be a set of intervals from $[\min F; \max F]$ contiguous to F and let $s_i(x)$ denote a distance from $x \in F$ to the interval $(a_i; a_i + \delta_i)$. Then convergence of (1) is equivalent to convergence of a series

$$\sum_i \left(\frac{\delta_i}{s_i(x)}\right)^{\lambda+1}. \tag{2}$$

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Thus, one can claim that the sequence $\left\{\frac{\delta_i}{s_i(x)}\right\}$ belongs to l_p for any $p > 1$ and almost all $x \in F$. I.Ya. Novikov conjectured that a stronger result takes place [2]. Recall that a *non-increasing rearrangement* of a real sequence $\{u_i\}_{i=1}^{\infty}$ is defined by

$$u_i^* := \inf\{\tau \geq 0 : \mathrm{card}\{j > 0 : |u_j| > \tau\} \leq i\},\ i = 1, 2, \dots$$

The rearrangement is well defined for any bounded sequence.

Conjecture 1.1 (Novikov). *For almost all $x \in F$ the sequence $\left\{\frac{\delta_i}{s_i(x)}\right\}$ is contained in $l_{1,\infty}$, that is,*

$$\sup_n n \left(\frac{\delta_n}{s_n(x)}\right)^* < \infty, \tag{3}$$

where $\left(\frac{\delta_n}{s_n(x)}\right)^*$ is the n-th element of the non-increasing rearrangement of $\left\{\frac{\delta_i}{s_i(x)}\right\}$.

Novikov suggested a stronger variant of the above conjecture known as the problem of intervals. Let $n > 0$ and $a_i, \delta_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, satisfy conditions

$$\delta_i > 0,\ i = 1, 2, \dots, n;\ a_1 \leq a_2 \leq \dots \leq a_n.$$

The sequence of intervals $\{[a_i; a_i + \delta_i]\}_{i=1}^n$ is then called *a configuration (of intervals).* Note that the elements of the configuration (intervals $[a_i; a_i + \delta_i]$) may intersect. The set of configurations for all possible values of n, a_i, δ_i will be denoted by $\mathcal{A}$. For a fixed configuration $v = \{[a_i; a_i + \delta_i]\}_{i=1}^n$ we can construct a set

$$\Omega(v) := \{t \in R : \exists k = k(t) > 0 : \mathrm{card}\{i : t \in [a_i; a_i + k\delta_i]\} \geq k\}, \tag{4}$$

where $\mathrm{card} M$ denotes the number of elements in the set M. In other words, a point $t \in R$ belongs to $\Omega(v)$ if and only if there exist $1 \leq k(t) \leq n$ and indexes $1 \leq i_1 < i_2 < \dots < i_{k(t)} \leq n$ such that t belongs to the intersection $\bigcap_{j=1}^{k(t)} [a_{i_j}; a_{i_j} + k(t)\delta_{i_j}]$. Evidently, the set $\Omega(v)$ is closed as a union of a finite number of closed sets. Let also

$$K(v) := \frac{\mu\Omega(v)}{\sum_{i=1}^n \delta_i}.$$

Next, for any set of configurations $B \subset \mathcal{A}$ define

$$K(B) := \sup\{c : c = K(v) \text{ for some configuration } v \in B\}.$$

Conjecture 1.2 (Novikov). *$K(\mathcal{A}) < \infty$.*

The problem of intervals is probably a difficult one, and we are far from getting a complete solution to it. Still, we are able to prove $K(B) < \infty$ for certain subsets $B \subset \mathcal{A}$. As a result, we will prove the Conjecture 1.1 for generalized Cantor sets.

2. Auxiliary results. We start with some elementary facts about sets Ω. The first lemma does not require a proof.

Lemma 2.1. *Let $v = \{[a_i; a_i + \delta_i]\}_{i=1}^n \in \mathcal{A}$, $d \in \mathbb{R}$ and $w = \{[a_i + d; a_i + d + \delta_i]\}_{i=1}^n$. Then*

$$\Omega(v) = \{t : t + d \in \Omega(w)\}.$$

Lemma 2.2. *For any configuration $v = \{[a_i; a_i + \delta_i]\}_{i=1}^n \in \mathcal{A}$ we have $[a_n; \max\Omega(v)] \subset \Omega(v)$.*

Proof. Since $\max\Omega(v) \in \Omega(v)$, there exist $1 \le k \le n$ and indexes $1 \le i_1 < i_2 < \dots < i_k \le n$ such that $\max\Omega(v) \le a_{i_j} + k\delta_{i_j}$ for $j = 1, 2, \dots, k$. It implies that for any $t \in [a_n; \max\Omega(v)]$ we have $a_{i_j} \le a_n \le t \le a_{i_j} + k\delta_{i_j}$, $j = 1, 2, \dots, k$, so that t belongs to $\Omega(v)$. ∎

LEMMA 2.3. *Let* $v = \{[a_i; a_i + \delta_i]\}_{i=1}^n \in \mathcal{A}$, $m \in [1; n-1]$ *and* $w = \{[a_i; a_i + \delta_i]\}_{i=1}^m$. *Then*

$$\Omega(w) = \Omega(v) \cap [a_1; \max\Omega(w)].$$

Proof. First, it is clear that $\Omega(w) \subset [a_1; \max\Omega(w)]$ and $\Omega(w) \subset \Omega(v)$. Now, let $t \in \Omega(v) \cap [a_1; \max\Omega(w)]$. If, additionally, $t \ge a_m$ then, by Lemma 2.2, $t \in \Omega(w)$. It is also evident that $\Omega(v) \cap [a_1; a_m) \subset \Omega(w)$. Combining the embeddings, we get the result. ∎

LEMMA 2.4. *Let* $v = \{[a_i; a_i + \delta_i]\}_{i=1}^n$ *be a configuration from* $\mathcal{A}$. *Then*

$$\max\Omega(v) - a_n \le \sum_{i=1}^n \delta_i.$$

Proof. Since $\max\Omega(v) \in \Omega(v)$, there exist $1 \le k \le n$ and $1 \le i_1 < i_2 < \dots < i_k \le n$, such that $\max\Omega(v) - a_{i_j} \le k\delta_{i_j}$ for all $1 \le j \le k$. It follows that

$$\max\Omega(v) - a_n \le \min_j(\max\Omega(v) - a_{i_j}) \le k \min_j \delta_{i_j} \le \sum_{i=1}^n \delta_i.$$

∎

Let $\mathcal{A}' := \{v \in \mathcal{A} : \text{the set } \Omega(v) \text{ is connected}\}$.

LEMMA 2.5. *Let* $v = \{[a_i; a_i + \delta_i]\}_{i=1}^n$ *be a configuration from* $\mathcal{A}$. *Then there exists a configuration* $w = \{[b_i; b_i + \delta_i]\}_{i=1}^n \in \mathcal{A}'$ *such that* $K(w) \ge K(v)$.

Proof. Suppose, $v \notin \mathcal{A}'$. Taking into account Lemma 2.2, there exists $m \in [1; n-1]$ such that $[a_1; a_m] \subset \Omega(v)$ but $[a_1; a_{m+1}] \not\subset \Omega(v)$. Let $x := \{[a_i; a_i + \delta_i]\}_{i=1}^m$. From Lemma 2.3 it follows that the set $\Omega(x)$ is connected. Indeed, since $[a_m; \max\Omega(x)] \subset \Omega(x) \subset \Omega(v)$, we have:

$$\Omega(x) = \Omega(v) \cap [a_1; \max\Omega(x)] = [a_1; a_m] \cup (\Omega(v) \cap [a_m; \max\Omega(x)]) = [a_1; \max\Omega(x)].$$

Let $a'_1, a'_2, \dots, a'_n$ be real numbers such that $a_i = a'_i$ for all $i = 1, 2, \dots, m$; $a'_i - a'_j = a_i - a_j$ for all $i, j \in \{m+1, \dots, n\}$; $a'_{m+1} = \max\Omega(\{[a_i; a_i + \delta_i]\}_{i=1}^m)$ and consider a configuration $v' = \{[a'_i; a'_i + \delta_i]\}_{i=1}^n$. By Lemma 2.3 and above reasoning, $[a'_1; a'_{m+1}] = \Omega(x) \subset \Omega(v')$. Further, let $t \in \Omega(v) \setminus \Omega(x)$ (note that it implies $t \ge a_{m+1}$). By definition, there exist $1 \le k(t) \le n$ and indexes $1 \le i_1 < i_2 < \dots < i_{k(t)} \le n$ such that $\max(a_{i_j}; \max\Omega(x)) \le t \le a_{i_j} + k(t)\delta_{i_j}$ for $j = 1, 2, \dots, k(t)$. Now, fix j and study two cases. If $i_j \in [m+1; n]$ then, clearly,

$$\max(a'_{i_j}; \max\Omega(x)) = a_{i_j} + a'_{m+1} - a_{m+1} < t + a'_{m+1} - a_{m+1} \le a'_{i_j} + k(t)\delta_{i_j}.$$

If $i_j \in [1; m]$ then with necessity $\max(a'_{i_j}; \max\Omega(x)) = a'_{m+1} \le t + a'_{m+1} - a_{m+1} \le a'_{i_j} + k(t)\delta_{i_j}$. Thus, $t + a'_{m+1} - a_{m+1} \in \Omega(v') \cap [\max\Omega(x); +\infty)$. It follows that $\mu(\Omega(v') \cap [\max\Omega(x); +\infty)) \ge \mu(\Omega(v) \setminus \Omega(x))$ whence $K(v') \ge K(v)$. Besides, if $m = n-1$ then $\Omega(v')$ is connected. Otherwise, we repeat the above arguments for configuration v' and construct a configuration $v'' = \{[a''_i; a''_i + \delta_i]\}_{i=1}^n$ such that $K(v'') \ge K(v')$ and $[a''_1; a''_{m+2}] \subset \Omega(v'')$.

If $m = n-2$ then the set $\Omega(v'')$ is connected and the lemma is proved. Otherwise, we apply the arguments once more, etc. ■

Next, fix $n > 0$ and consider a function

$$f : \{1, 2, \dots, n-1\} \to 2^{\{1,2,\dots,n-1\}}$$

where $2^{\{1,2,\dots,n-1\}}$ denotes the set of all possible subsets of $\{1, 2, \dots, n-1\}$. We will call f *the function of intervals (of order n)* if for all $1 \le i \le n-1$ the set $f(i)$ is not empty and $\max\{j : j \in f(i)\} = i$. Now, let $v = \{[a_i; a_i + \delta_i]\}_{i=1}^n \in \mathcal{A}$ and f be some function of intervals. We will say that the configuration v is *f-admissible* if

$$a_{i+1} \le \max \Omega(\{[a_j; a_j + \delta_j]\}_{j \in f(i)}) \tag{5}$$

for $i = 1, 2, \dots, n-1$. Whenever for all $1 \le i \le n-1$ we have equality in (5), the configuration v is called *f-optimal.*

Remark. Clearly, any f-admissible configuration belongs to $\mathcal{A}'$.

Remark. It is not difficult to see that if we define a function of intervals g of order n by $g(i) = \{1, 2, \dots, i\}$, $i = 1, 2, \dots, n-1$, then any configuration $v = \{[a_i; a_i + \delta_i]\}_{i=1}^n \in \mathcal{A}'$ is g-admissible. Indeed, since the set $\Omega(v)$ is connected, $[a_i; a_{i+1}) \subset \Omega(v)$, implying $[a_i; a_{i+1}) \subset \Omega(\{[a_j; a_j + \delta_j]\}_{j=1}^i)$, $i = 1, 2, \dots, n-1$. So, $a_{i+1} \le \max \Omega(\{[a_j; a_j + \delta_j]\}_{j=1}^i)$, $i = 1, 2, \dots, n-1$.

Lemma 2.6. *Let f be some function of intervals of order n and $w = \{[b_i; b_i + \delta_i]\}_{i=1}^n$, $v = \{[a_i; a_i + \delta_i]\}_{i=1}^n$ be an f-optimal and an f-admissible configuration, respectively. Then $K(w) \ge K(v)$.*

Proof. By Lemma 2.1, it is sufficient to check the case $b_1 = a_1$. We need to prove that $\max \Omega(w) \ge \max \Omega(v)$. Let $m \in [1; n]$ be the maximum number such that $b_i \ge a_i$, $i = 1, 2, \dots, m$. Suppose that $m \le n-1$ (note that $a_{m+1} \ge b_m$ because otherwise $b_{m+1} > a_{m+1}$, and the assumption is wrong). By definition of f-admissibility, there exist $1 \le k \le \operatorname{card} f(m)$ and $i_1 < i_2 < \dots < i_k$, $i_l \in f(m)$, $l = 1, 2, \dots, k$ such that a_{m+1} belongs to the intersection of intervals $[a_{i_l}; a_{i_l} + k\delta_{i_l}]$, $l = 1, 2, \dots, k$. Since $a_{m+1} \ge b_m$ and $b_{i_l} \ge a_{i_l}$, $l = 1, 2, \dots, k$, the point a_{m+1} belongs to $\bigcap_{l=1}^{k} [b_{i_l}; b_{i_l} + k\delta_{i_l}]$. It follows that $a_{m+1} \le \max \Omega(\{[b_j; b_j + \delta_j]\}_{j \in f(m)}) = b_{m+1}$ and the assumption is wrong. Thus, for all $i \in [1; n]$ we have $b_i \ge a_i$. Finally, applying similar arguments to $\max \Omega(v)$, we get that $\max \Omega(w) \ge \max \Omega(v)$. ■

Further, we will need one more definition. Let $v = \{[a_i; a_i + \delta_i]\}_{i=1}^n$ be an f-admissible configuration for some function of intervals f of order n and $C > 0$ be some constant. Then by *C, f-rarefication* of v we will mean any configuration $w = \{[b_i; b_i + \kappa_i]\}_{i=1}^{2n-1}$, where $\kappa_{2i-1} = \delta_i$ for $i = 1, 2, \dots, n$, $\sum_{i=1}^{n-1} \kappa_{2i} \le C \sum_{i=1}^{n} \delta_i$ and points b_i are defined as follows:

$$b_{2i} = b_{2i-1} + \kappa_{2i-1}, \ i = 1, 2, \dots, n-1;$$
$$b_{2i+1} = \max(b_{2i} + \kappa_{2i}, \max \Omega(\{[b_{2j-1}; b_{2j-1} + \kappa_{2j-1}]\}_{j \in f(i)})), \ i = 1, 2, \dots, n-1.$$

It is not difficult to see that $w \in \mathcal{A}'$, that is, $\Omega(w)$ is connected. Indeed, $[b_{2i-1}; b_{2i}] = [b_{2i-1}; b_{2i-1} + \kappa_{2i-1}] \subset \Omega(w)$, $i = 1, 2, \ldots, n-1$. Further, $[b_{2i}; b_{2i} + \kappa_{2i}] \subset \Omega(w)$ and, by Lemma 2.2,

$$[b_{2i-1}; \max \Omega(\{[b_{2j-1}; b_{2j-1} + \kappa_{2j-1}]\}_{j \in f(i)})] \subset \\ \Omega(\{[b_{2j-1}; b_{2j-1} + \kappa_{2j-1}]\}_{j \in f(i)}) \subset \Omega(w)$$

whence $[b_{2i}; b_{2i+1}] \subset \Omega(w)$, $i = 1, 2, \ldots, n-1$. Finally, $[b_{2n-1}; \max \Omega(w)] \subset \Omega(w)$ (by Lemma 2.2) and we get $[b_1; \max \Omega(w)] = \Omega(w)$.

Note, that there exist continuum rarefications for any given configuration. Still, there is a common property shared by all of them.

LEMMA 2.7. *Let $v = \{[a_i; a_i + \delta_i]\}_{i=1}^{n}$ be an f-admissible configuration for a function of intervals f and $w = \{[b_i; b_i + \kappa_i]\}_{i=1}^{2n-1}$ be a C, f-rarefication of v. Then $\mu\Omega(w) \geq \mu\Omega(v)$.*

Proof. By Lemma 2.1, it suffices to consider the case $a_1 = b_1$. We need to prove that $b_{2i-1} \geq a_i$, $i = 1, 2, \ldots, n$; then we will automatically get $\max \Omega(w) \geq \max \Omega(v)$. We will act as in proof of Lemma 2.6. Let $m \in [1; n]$ be the maximum number such that $b_{2i-1} \geq a_i$, $i = 1, 2, \ldots, m$ and suppose that $m \leq n - 1$. By definition of C, f-rarefications, $b_{2m+1} \geq \max \Omega(\{[b_{2j-1}; b_{2j-1} + \kappa_{2j-1}]\}_{j \in f(m)})$. On the other hand, $a_{m+1} \leq \max \Omega(\{[a_j; a_j + \delta_j]\}_{j \in f(m)})$, where $\delta_j = \kappa_{2j-1}$ and $a_j \leq b_{2j-1}$, $j = 1, 2, \ldots, m$. It follows that $b_{2m+1} \geq a_{m+1}$. Consequently, the assumption is wrong and $m = n$. ∎

REMARK. With v and w defined as in Lemma 2.7, it is clear that $K(w) \geq \frac{1}{C+1} K(v)$.

LEMMA 2.8. *Let f be a function of intervals of order n, $w = \{[b_i; b_i + \kappa_i]\}_{i=1}^{2n-1}$ be a C, f-rarefication of an f-admissible configuration v and $\tilde{w} = \{[\tilde{b}_i; \tilde{b}_i + \kappa_i]\}_{i=1}^{2n-1}$ be a configuration from $\mathcal{A}'$ such that*

$$\tilde{b}_{2i} = \tilde{b}_{2i-1} + \kappa_{2i-1}, \ i = 1, 2, \ldots, n-1;$$
$$\tilde{b}_{2i+1} \leq \max(\tilde{b}_{2i} + \kappa_{2i}, \max \Omega(\{[\tilde{b}_{2j-1}; \tilde{b}_{2j-1} + \kappa_{2j-1}]\}_{j \in f(i)})), \ i = 1, 2, \ldots, n-1.$$

Then $\mu\Omega(\tilde{w}) \leq \mu\Omega(w)$.

Proof. The proof can be conducted the same way as in Lemma 2.6: we suppose that $b_1 = \tilde{b}_1$ and show by induction that $b_i \geq \tilde{b}_i$, $i \in [1; 2n-1]$. It follows that $\max \Omega(w) \geq \max \Omega(\tilde{w})$. ∎

Further, we will use the notation *C-rarefication* instead of C, f-rarefication when $f(i) = \{1, 2, \ldots, i\}$, $i = 1, 2, \ldots$ The essence of the conception of rarefications is that in some cases it is much simpler to estimate the measure of Ω for a rarefication than for the original configuration. Next, we will prove Conjecture 1.2 with an additional restriction on lengths of intervals. Let $\mathcal{M} \subset \mathcal{A}$ be a set of configurations $\{[a_i; a_i + \delta_i]\}_{i=1}^{n}$ defined for all possible n, a_i and δ_i satisfying $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$.

THEOREM 2.9. $K(\mathcal{M}) < \infty$.

Proof. By Lemma 2.5, it is sufficient to prove the statement for configurations $v = \{[a_i; a_i + \delta_i]\}_{i=1}^{n} \in \mathcal{M} \cap \mathcal{A}'$. Let

$$n(k) = \operatorname{card}\{i \in [1; n] : 4^k \leq \delta_i < 4^{k+1}\}, \ k \in \mathbb{Z}.$$

Next, we construct a 256-rarefication $w = \{[b_i; b_i + \kappa_i]\}_{i=1}^{2n-1}$ of v by setting

$$\kappa_{2i} = 32 \max\{4^l : l \in \mathbb{Z},\ \sum_{k=l}^{+\infty} 2^{k-l} n(k) \geq i\},\ i = 1, 2, \ldots, n-1. \tag{6}$$

Note that the function $N(l) = \sum_{k=l}^{+\infty} 2^{k-l} n(k)$ is non-increasing, vanishing for sufficiently large l, and $\lim_{l \to -\infty} N(l) = +\infty$, so, κ_{2i} are determined correctly. Further, for any $l \in \mathbb{Z}$

$$\{i \in [1; n-1] : \kappa_{2i} \geq 32 \cdot 4^l\} = \{i \in [1; n-1] : \max\{4^p : N(p) \geq i\} \geq 4^l\} = \{i : 1 \leq i \leq \min(n-1; N(l))\}$$

and we get an estimate

$$\sum_{i=1}^{n-1} \kappa_{2i} \leq 4 \sum_{l=-\infty}^{+\infty} 32 \cdot 4^l \mathrm{card}\{i \in [1; n-1] : \kappa_{2i} \geq 32 \cdot 4^l\} \leq 128 \sum_{l=-\infty}^{+\infty} 4^l N(l) =$$
$$128 \sum_{l=-\infty}^{+\infty} 2^l \sum_{k=l}^{+\infty} 2^k n(k) = 128 \sum_{k=-\infty}^{+\infty} 2^k n(k) \sum_{l=-\infty}^{k} 2^l = 256 \sum_{k=-\infty}^{+\infty} 4^k n(k) \leq 256 \sum_{i=1}^{n} \delta_i,$$

so w is indeed a rarefication with constant 256.

Our next goal is to show that $b_{2i+1} = b_{2i} + \kappa_{2i}$ for all $1 \leq i \leq n-1$. Fix $i \in [1; n-1]$ and consider any $j \in [1; i]$. Let $p \in \mathbb{Z}$ be such a number that $4^p \leq \kappa_{2j-1} < 4^{p+1}$. By definition of $\mathcal{M}$, the sequence $\{\kappa_{2h-1}\}_{h=1}^{n}$ is non-increasing and $\sum_{k=p}^{+\infty} n(k) = \mathrm{card}\{h \in [1; n] : \kappa_{2h-1} \geq 4^p\} \geq j$. Therefore, $N(p-1) \geq \sum_{k=p}^{+\infty} 2n(k) \geq 2j$. Denote by α_j the *relative length* of the interval $[b_{2j-1}; b_{2j-1} + \kappa_{2j-1}]$ with respect to the distance between b_{2i+1} and b_{2j-1}, i.e the value $\frac{\kappa_{2j-1}}{b_{2i+1} - b_{2j-1}}$.

First, suppose that $2j \geq i$. Then $N(p-1) \geq i$ and for all $h \in [j; i]$

$$\kappa_{2h} \geq 32 \max\{4^l : N(l) \geq i\} \geq 32 \cdot 4^{p-1} > 2\kappa_{2j-1}. \tag{7}$$

Applying (7) we get an estimation

$$\alpha_j \leq \frac{\kappa_{2j-1}}{b_{2i} + \kappa_{2i} - b_{2j-1}} \leq \frac{\kappa_{2j-1}}{\sum_{h=j}^{i} \kappa_{2h}} < \frac{1}{2(i-j+1)}. \tag{8}$$

Next, consider the case $2j < i$. Using similar arguments, we get for all $h \in [j; 2j]$

$$\kappa_{2h} = 32 \max\{4^l : N(l) \geq h\} \geq 32 \max\{4^l : N(l) \geq 2j\} > 2\kappa_{2j-1}.$$

Thus,

$$\alpha_j \leq \frac{\kappa_{2j-1}}{b_{2i} + \kappa_{2i} - b_{2j-1}} \leq \frac{\kappa_{2j-1}}{\sum_{h=j}^{2j} \kappa_{2h}} < \frac{1}{2(j+1)}. \tag{9}$$

Now, suppose that $b_{2i+1} = \max \Omega(\{[b_{2h-1}; b_{2h-1} + \kappa_{2h-1}]\}_{h=1}^{i}) > b_{2i} + \kappa_{2i}$. It means that there exist $k \in [1; i]$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq i$ such that $b_{2i+1} \leq b_{2i_l - 1} + k\kappa_{2i_l - 1}$, that is, $\alpha_{i_l} \geq \frac{1}{k}$, $l = 1, 2, \ldots, k$. On the other hand, from (8) and (9) it follows that

$\alpha_h < \frac{1}{2\min(i-h+1;h+1)}$, $h = 1, 2, \ldots, i$. Consequently,

$$k \le \operatorname{card}\{h \in [1; i] : k\alpha_h \ge 1\} \le \operatorname{card}\{h \in [1; i] : 2\min(i-h+1; h+1) < k\} \le \\ \operatorname{card}\{h \in [1; i] : 2(i-h+1) < k\} + \operatorname{card}\{h \in [1; i] : 2(h+1) < k\} < k$$

and the assumption is wrong. Thus, $b_{2i+1} = b_{2i} + \kappa_{2i}$ for all $i \in [1; n-1]$. It remains to note that, according to Lemma 2.4, $\max \Omega(w) - b_{2n-1} \le \sum_{j=1}^{2n-1} \kappa_j$. Finally, by Lemma 2.7,

$$\mu\Omega(v) \le \mu\Omega(w) = \sum_{j=1}^{2n-2} \kappa_j + (\max \Omega(w) - b_{2n-1}) < 2\sum_{j=1}^{2n-1} \kappa_j \le 2(256+1)\sum_{j=1}^{n} \delta_j,$$

and $K(v) \le 514$. ■

3. Generalized Cantor sets. First, let us clarify the notion "generalized Cantor set". Let $[a; b]$ be some interval and $\{d_i\}_{i=1}^{\infty}$ be a sequence of positive reals, such that

$$\sum_{i=1}^{\infty} 2^{i-1} d_i < b - a.$$

We construct a sequence of open intervals $\{I_i\}_{i=1}^{\infty}$ by induction. At first step, let I_1 be an open interval of length d_1 from $[a; b]$ such that $a < \inf I_1$ and $\sup I_1 < b$. We will refer to it as the interval *of the first rank*. At second step, we choose two intervals *of the second rank* I_2 and I_3 such that $\mu I_2 = \mu I_3 = d_2$ and $a < \inf I_2 < \sup I_2 < \inf I_1 < \sup I_1 < \inf I_3 < \sup I_3 < b$. At step k, $k > 2$, we have $2^{k-1} - 1$ intervals already constructed. The set $[a; b] \setminus \bigcup_{i=1}^{2^{k-1}-1} I_i$ consists of 2^{k-1} closed segments $K_{k,1}, K_{k,2}, \ldots, K_{k,2^{k-1}}$ (numbered from left to right). Then for each $i \in [2^{k-1}; 2^k - 1]$ we choose $I_i \subset K_{k,i-2^{k-1}+1}$ *of rank* k having length d_k such that $\inf K_{k,i-2^{k-1}+1} < \inf I_i$ and $\sup I_i < \sup K_{k,i-2^{k-1}+1}$ (certainly, we must see to it that $K_{k,j}$ are large enough to comprise corresponding open intervals). Let $\mathcal{C} := [a; b] \setminus \bigcup_{i=1}^{\infty} I_i$. Whenever $\mathcal{C}$ is nowhere dense, we will call it *the generalized Cantor set*. It is clear that $\mathcal{C}$ is closed and of positive measure.

Next, for a given $n > 0$ consider a function $r_n = r_n(i)$ defined on the set $\{1, 2, \ldots, 2^n - 1\}$ by the formula

$$r_n(i) = n - \max\{k \ge 0 : i \equiv 0 \mod 2^k\}. \tag{10}$$

In other words, $r_n(i)$ is the minimal number such that $i2^{r_n(i)-n}$ is an integer (certainly, $i2^{r_n(i)-n}$ is odd). Clearly, r_n takes its values from $\{1, 2, \ldots, n\}$ and for each $k \in \{1, 2, \ldots, n\}$ we have $\operatorname{card}\{h \in [1; 2^n - 1] : r_n(h) = k\} = 2^{k-1}$. Let also

$$q_n(i) := 2^{r_n(i)-1} - 1 + \operatorname{card}\{h \in [1; i] : r_n(h) = r_n(i)\}$$

for $i \in [1; 2^n - 1]$. Note that the value $2^{r_n(i)-1} - 1$ is the number of indexes h such that $r_n(h) < r_n(i)$. It is not difficult to see that q_n is a bi-unique mapping of $\{1, 2, \ldots, 2^n - 1\}$ onto itself. Specifically, all $h \in [1; 2^n - 1]$ with $r_n(h) = k$, $k \in [1; n]$, are mapped into $[2^{k-1}; 2^k - 1]$.

Next, let d_i, $i = 1, 2, \ldots, n$, be some positive reals. A set of configurations s defined by the formula

$$s = \{[a_i; a_i + d_{r_n(i)}]\}_{i=1}^{2^n-1} \tag{11}$$

for all possible n, a_i, d_i will be denoted by $\mathcal{T}$. Note that, in the above construction of $\mathcal{C}$, $\{\overline{I}_{q_n(i)}\}_{i=1}^{2^n-1} \in \mathcal{T}$ for any $n > 0$ where $\overline{I}_{q_n(i)}$ is the closure of $I_{q_n(i)}$ (we use the function q_n here to reorder the intervals $\{\overline{I}_i\}_{i=1}^{2^n-1}$ to get the valid configuration). Moreover, for any $i \in [1; 2^n - 1]$ the rank of $I_{q_n(i)}$ coincides with the value of $r_n(i)$. So, it is natural to refer to the i-th interval of a configuration $s = \{[a_i; a_i + \delta_i]\}_{i=1}^{2^n-1} \in \mathcal{T}$ as one having *the rank* $r_n(i)$. It is clear that there is a straightforward connection between generalized Cantor sets and configurations from $\mathcal{T}$, so the latter will be used as a tool for proving the property we are concerned with. Now, we are ready to formulate the main results of the paper.

Theorem 3.1. $K(\mathcal{T}) < \infty$.

An immediate consequence of the above theorem is the following

Corollary 3.2. *Let $\mathcal{C}$ be an arbitrary generalized Cantor set, $\{d_i\}_{i=1}^{\infty}$ and $\{I_i\}_{i=1}^{\infty}$ have the same meaning as in the definition of $\mathcal{C}$ and $s_i(x)$ denote the distance from x to the interval I_i. Then for almost all points $x \in \mathcal{C}$*

$$\left\{\frac{\mu I_i}{s_i(x)}\right\}_{i=1}^{\infty} \in l_{1,\infty}.$$

Proof. For any $m > 0$ let

$$\Omega_m := \{t : \exists k = k(t) > 0 : \operatorname{card}\{i \geq 2^{m-1} : t \in (\inf I_i; \inf I_i + 2k\mu I_i)\} \geq k\}$$

and

$$\Omega_{-m} := \{t : \exists k = k(t) > 0 : \operatorname{card}\{i \geq 2^{m-1} : t \in (\sup I_i - 2k\mu I_i; \sup I_i)\} \geq k\}.$$

First, it is clear that $\Omega_m = \bigcup\limits_{h=m+1}^{\infty} \Omega_{m,h}$, where

$$\Omega_{m,h} := \{t : \exists k = k(t) > 0 : \operatorname{card}\{i \in [2^{m-1}; 2^{h-1} - 1] : t \in (\inf I_i; \inf I_i + 2k\mu I_i)\} \geq k\}.$$

Besides, $\Omega_{m,h} \subset \Omega_{m,h+1}$, $h = m + 1, m + 2, \ldots$, so by continuity of measure $\mu\Omega_m = \lim\limits_{h} \Omega_{m,h}$. On the other hand, $\Omega_{m,h} \subset \Omega(s)$ for any configuration $s = \{[a_i; a_i+\delta_i]\}_{i=1}^{2^h-1} \in \mathcal{T}$ satisfying conditions $a_i = \inf I_{q_h(i)}$, $\delta_i = 2\mu I_{q_h(i)}$ for $i \in \{l : r_h(l) \geq m\}$ (note that for other indexes we can choose as small δ_i as we please). In view of Theorem 3.1, $\mu\Omega(s) \leq K(\mathcal{T}) \left(\epsilon + \sum\limits_{i=2^{m-1}}^{2^{h-1}-1} 2\mu I_i\right)$ where $\epsilon = \sum\limits_{i:r_h(i)<m} \delta_i$ can be as small as we want. Consequently, $\mu\Omega_m \leq 2K(\mathcal{T}) \sum\limits_{i=2^{m-1}}^{\infty} \mu I_i$. In the definition of Ω_{-m}, the participating intervals "expand" in the opposite direction, nevertheless, we can apply the above arguments to the set $\{t : -t \in \Omega_{-m}\}$. So, we get $\mu\Omega_{-m} \leq 2K(\mathcal{T}) \sum\limits_{i=2^{m-1}}^{\infty} \mu I_i$.

It is not difficult to see that for any point $x \notin \Omega_m \cup \Omega_{-m}$

$$\left\{ \frac{\mu I_{i+2^{m-1}-1}}{s_{i+2^{m-1}-1}(x)} \right\}_{i=1}^{\infty} \in l_{1,\infty}. \tag{12}$$

Indeed, suppose that (12) is false. Then, in particular, there exist $k > 0$ and $0 < i_1 < i_2 < \dots < i_{2k}$ such that $\frac{\mu I_{i_l+2^{m-1}-1}}{s_{i_l+2^{m-1}-1}(x)} > \frac{1}{k}$, $l = 1, 2, \dots, 2k$. This implies

$$x \in (\inf I_{i_l+2^{m-1}-1} - k\mu I_{i_l+2^{m-1}-1}; \sup I_{i_l+2^{m-1}-1} + k\mu I_{i_l+2^{m-1}-1}) \subset \\ (\sup I_{i_l+2^{m-1}-1} - 2k\mu I_{i_l+2^{m-1}-1}; \inf I_{i_l+2^{m-1}-1} + 2k\mu I_{i_l+2^{m-1}-1}),$$

$l = 1, 2, \dots, 2k$, and, consequently, $x \in \Omega_m \cup \Omega_{-m}$. Thus, the assumption is wrong and (12) is fulfilled. It follows easily that for almost all points $x \in \mathcal{C} \setminus (\Omega_m \cup \Omega_{-m})$,

$$\left\{ \frac{\mu I_i}{s_i(x)} \right\}_{i=1}^{\infty} \in l_{1,\infty}.$$

Taking m to infinity and having in mind the estimations for $\mu\Omega_m$, $\mu\Omega_{-m}$, we get the result. ■

Before proving Theorem 3.1, we need to consider some auxiliary statements.

Let us begin by making some notations. Let $n > 0$. Every $i \in \{1, 2, \dots, 2^n - 1\}$ can be uniquely represented in the form

$$i = \sum_{j=0}^{n-1} \gamma_j 2^j, \quad \gamma_j \in \{0; 1\},\ 0 \le j \le n-1. \tag{13}$$

Define a function of intervals $P_n : \{1, 2, \dots, 2^n - 2\} \to 2^{\{1,2,\dots,2^n-2\}}$ by the formula

$$P_n(i) = \left\{ h : \exists k \in [0; n-1] \text{ such that } h = \sum_{j=k}^{n-1} \gamma_j 2^j \right\},$$

where γ_j are coefficients from the "binary" representation of i (13). Note that each element of $P_n(i)$ is the maximal multiple of 2^k (for some $k \ge 0$) not exceeding i. Cardinality of $P_n(i)$ equals to the number of "1"-s in the "binary" representation of i.

Further, let f be some function of intervals of order $2^n - 1$ and s be an f-admissible configuration from $\mathcal{T}$ defined by the formula (11). Then the C, f-rarefication $s' = \{b_i; b_i + \kappa_i\}_{i=1}^{2^{n+1}-3}$ of the configuration s is called *uniform* if for all $1 \le i \le 2^n - 2$ we have

$$\kappa_{2i} = \frac{C}{2^n - 2} \sum_{j=1}^{2^n-1} d_{r_n(j)}.$$

Lemma 3.3. *Let $s \in \mathcal{T}$ be a configuration defined by* (11) *and $s' = \{[a'_i; a'_i + 2d_{r_n(i)}]\}_{i=1}^{2^n-1}$ be a P_n-admissible configuration. Then for any $C_1 \ge 2$ and the uniform C_1, P_n-rarefication w' of s' we have $\mu\Omega(w') \ge \mu\Omega(s)$.*

Proof. According to Lemma 2.5, it suffices to consider configurations $s \in \mathcal{T} \cap \mathcal{A}'$. Let $\delta_i = d_{r(i)}$, $1 \le i \le 2^n - 1$. We may assume that $n > 1$. Consider the uniform C-rarefication $w = \{[b_i; b_i + \kappa_i]\}_{i=1}^{2^{n+1}-3}$ of the configuration s ($C \ge 4$). Suppose that $b_{2i+1} > b_{2i} + \kappa_{2i}$

for some $i \in [1; 2^n - 2]$ and let $M_i \subset \{1, 2, \dots, i\}$ be a set of indexes such that

$$b_{2i+1} = \max \Omega(\{[b_{2j-1}; b_{2j-1} + \kappa_{2j-1}]\}_{j=1}^{i}) = \max \bigcap_{j \in M_i} [b_{2j-1}; b_{2j-1} + \mathrm{card} M_i \cdot \kappa_{2j-1}]. \quad (14)$$

Consider two cases.

1) $\mathrm{card}(M_i \cap P_n(i)) \geq \frac{1}{2}\mathrm{card} M_i$. Then, clearly,

$$b_{2i+1} \leq \max \bigcap_{j \in M_i \cap P_n(i)} [b_{2j-1}; b_{2j-1} + 2\mathrm{card}(M_i \cap P_n(i)) \cdot \kappa_{2j-1}]. \quad (15)$$

2) $\mathrm{card}(M_i \cap P_n(i)) < \frac{1}{2}\mathrm{card} M_i$. First, note that whenever $k \in M_i \setminus P_n(i)$, there exists $l \in \{k+1, k+2, \dots, i\}$ such that $r_n(l) \leq r_n(k)$. Indeed, suppose it is not true. Then, by definition of the function r_n, none of the numbers $k+1, k+2, \dots, i$ is a multiple of $2^{n-r_n(k)}$. Thus, k is the maximal multiple of $2^{n-r_n(k)}$ not exceeding i, so $k \in P_n(i)$ and the assumption is wrong. Further, if $k \in M_i \setminus P_n(i)$ and $r_n(l) \leq r_n(k)$ for some $l \in [k+1; i]$ then $l - k \geq 2^{n-r_n(k)}$, so

$$b_{2l-1} - b_{2k-1} \geq 2^{n-r_n(k)} \frac{C}{2^n - 2} \sum_{j=1}^{2^n-1} \delta_j. \quad (16)$$

For some $r \in [1; n]$ consider a set

$$Z_r := \{k \in M_i \setminus P_n(i) : r_n(k) = r\}.$$

Suppose that $\mathrm{card} Z_r > 0$ and denote elements of Z_r by i_h, $h \in [1; \mathrm{card} Z_r]$, $i_1 < i_2 < \dots < i_{\mathrm{card} Z_r}$. In view of (14) and (16),

$$\mathrm{card} M_i \cdot d_r > b_{2i-1} - b_{2i_1-1} = (b_{2i-1} - b_{2i_{\mathrm{card} Z_r}-1}) + \sum_{h=1}^{\mathrm{card} Z_r - 1} (b_{2i_{h+1}-1} - b_{2i_h-1}) \geq 2^{n-r} \mathrm{card} Z_r \frac{C}{2^n - 2} \sum_{j=1}^{2^n-1} \delta_j,$$

whence

$$\mathrm{card} Z_r < 4 \frac{2^{r-1} d_r \mathrm{card}(M_i \setminus P_n(i))}{C \sum\limits_{j=1}^{n} 2^{j-1} d_j}.$$

Summing by all r such that $\mathrm{card} Z_r > 0$ we get

$$\mathrm{card}(M_i \setminus P_n(i)) < \frac{4}{C} \mathrm{card}(M_i \setminus P_n(i)).$$

Since $C \geq 4$, the last inequality is impossible, implying $\mathrm{card}(M_i \cap P_n(i)) \geq \frac{1}{2}\mathrm{card} M_i$.

Thus, by (15), for all $1 \leq i \leq 2^n - 2$

$$b_{2i+1} \leq \max\{b_{2i} + \kappa_{2i}, \max \Omega(\{[b_{2j-1}; b_{2j-1} + 2\kappa_{2j-1}]\}_{j \in P_n(i)})\}. \quad (17)$$

Now, let $\tilde{w} = \{[b_i; b_i + \tilde{\kappa}_i]\}_{i=1}^{2^{n+1}-3}$ with $\tilde{\kappa}_{2i-1} = 2\kappa_{2i-1}$, $i \in [1; 2^n - 1]$; $\tilde{\kappa}_{2i} = \kappa_{2i}$, $i \in [1; 2^n - 2]$ and let $w' = \{[b'_i; b'_i + \kappa'_i]\}_{i=1}^{2^{n+1}-3}$ be the uniform $(C/2), P_n$-rarefication of s'. Clearly, $\mu\Omega(\tilde{w}) \geq \mu\Omega(w)$. Further, $\tilde{\kappa}_i = \kappa'_i$ for all $1 \leq i \leq 2^{n+1} - 3$ and from (17) and

Lemma 2.8, $\mu\Omega(\tilde{w}) \leq \mu\Omega(w')$. Thus,

$$\mu\Omega(w') \geq \mu\Omega(\tilde{w}) \geq \mu\Omega(w) \geq \mu\Omega(s).$$

■

Lemma 3.3 will allow us to consider only P_n-admissible configurations and their rarefications when proving Theorem 3.1. However, the rarefications are not so simple as in proof of Theorem 2.9, and we have to use some probabilistic results to estimate $K(\mathcal{T})$. The following lemma delivers an upper estimate for the probability of a specific event for the Bernoulli process. It is quite possible that the result cannot be directly derived from the known facts, so, the full proof is given.

LEMMA 3.4. *Let $\tilde{\eta} = \{\eta_j\}_{j=1}^{\rho}$ be a sequence of independent random variables η_j, $P\{\eta_j = 1\} = P\{\eta_j = 0\} = \frac{1}{2}$. Let also $k, l_1, l_2, \dots, l_\rho$ be some non-negative integers such that*

$$\sum_{j=1}^{\rho} 2^{-l_j} \leq \frac{k}{e}.$$

Finally, let

$$G_{\rho,j}(\tilde{\eta}) = \begin{cases} 1, & \text{if } j \leq \rho - l_j, \eta_j = 1 \text{ and } \eta_v = 0 \text{ for all } v \in [j+1; j+l_j], \\ 0, & \text{else.} \end{cases}$$

Then

$$P\{\sum_{j=1}^{\rho} G_{\rho,j}(\tilde{\eta}) \geq k\} \leq 2^{-k} \tag{18}$$

Proof. Note that $P\{G_{\rho,j}(\tilde{\eta}) = 1\} \in \{0; 2^{-l_j-1}\}$, $j = 1, 2, \dots, \rho$. Now, let $1 \leq j_1 < j_2 < \dots < j_k \leq \rho$. It is evident that whenever $j_v + l_v \geq j_{v+1}$ for some $1 \leq v < k$ or $j_k > \rho - l_k$ then

$$P\{G_{\rho,j_1}(\tilde{\eta}) = G_{\rho,j_2}(\tilde{\eta}) = \dots = G_{\rho,j_k}(\tilde{\eta}) = 1\} = 0.$$

Otherwise, the events $(G_{j_v}(\tilde{\eta}) = 1)$, $v = 1, 2, \dots, k$, are independent. Thus, we have

$$P\{G_{\rho,j_1}(\tilde{\eta}) = G_{\rho,j_2}(\tilde{\eta}) = \dots = G_{\rho,j_k}(\tilde{\eta}) = 1\} \leq 2^{-k-\sum\limits_{v=1}^{k} l_{j_v}}.$$

As a consequence we get the following estimate:

$$P\{\sum_{j=1}^{\rho} G_{\rho,j}(\tilde{\eta}) \geq k\} \leq \sum_{j_1, j_2, \dots, j_k} 2^{-k-\sum\limits_{v=1}^{k} l_{j_v}}, \tag{19}$$

where the sum in the right part is taken over all samples $1 \leq j_1 < j_2 < \dots < j_k \leq \rho$. Let us denote the right part of the inequality (19) by $h(l_1, l_2, \dots, l_\rho)$. We will consider h as a function of real variables $x_1, x_2, \dots, x_\rho$ with the restrictions

$$\sum_{j=1}^{\rho} 2^{-x_j} \leq \frac{k}{e}. \tag{20}$$

We will show that h reaches its maximum when $x_1 = x_2 = \cdots = x_\rho$. Indeed, suppose that $\mathrm{argmax} h = (x'_1, x'_2, \ldots, x'_\rho)$, $x'_n \neq x'_m$ for some n and m. Let

$$x''_j = \begin{cases} x'_j, & j \notin \{n,m\}, \\ 1 - \log_2(2^{-x'_n} + 2^{-x'_m}), & j \in \{n,m\}. \end{cases}$$

Clearly,

$$2^{-x''_n} + 2^{-x''_m} = 2^{-x'_n} + 2^{-x'_m}, \tag{21}$$

so, (20) is fulfilled for x''_j. Further, in view of (21),

$$h(x''_1, x''_2, \ldots, x''_\rho) - h(x'_1, x'_2, \ldots, x'_\rho) = \sum_{j_1, j_2, \ldots, j_k} \left(2^{-k - \sum\limits_{v=1}^{k} x''_v} - 2^{-k - \sum\limits_{v=1}^{k} x'_v} \right) =$$

$$\sum_{\substack{j_1, j_2, \ldots, j_{k-2} \\ n, m \notin \{j_1, \ldots, j_{k-2}\}}} (2^{-x''_n - x''_m} - 2^{-x'_n - x'_m}) 2^{-k - \sum\limits_{v=1}^{k-2} x'_{j_v}},$$

where the last sum is taken over all samples $1 \leq j_1 < j_2 < \cdots < j_{k-2} \leq \rho$ such that $n, m \notin \{j_1, j_2, \ldots, j_{k-2}\}$. Taking into account that

$$2^{-x''_n - x''_m} = 2^{2\log_2(2^{-x'_n} + 2^{-x'_m}) - 2} = \frac{1}{4}(2^{-x'_n} + 2^{-x'_m})^2 =$$

$$\frac{1}{4}(2^{-x'_n} - 2^{-x'_m})^2 + 2^{-x'_n - x'_m} > 2^{-x'_n - x'_m},$$

we get $h(x''_1, x''_2, \ldots, x''_\rho) > h(x'_1, x'_2, \ldots, x'_\rho)$. Thus, the assumption was false, and

$$x'_1 = x'_2 = \cdots = x'_\rho = \log_2 \frac{e\rho}{k}.$$

Finally, applying the Stirling formula, we obtain

$$h(x'_1, x'_2, \ldots, x'_\rho) = \binom{k}{\rho} 2^{-k(1 + \log_2 \frac{e\rho}{k})} = 2^{-k} \frac{\rho!}{k!(\rho-k)!} \left(\frac{k}{e\rho}\right)^k \leq 2^{-k} \frac{k^k}{k! e^k} \leq 2^{-k},$$

and (18) follows. ∎

Proof of Theorem 3.1. Certainly, we may assume that $n > 1$. Let s be an arbitrary configuration from $\mathcal{T}$ defined by the formula (11), and $s' = \{[a'_i; a'_i + 2d_{r_n(i)}]\}_{i=1}^{2^n - 1}$ be a P_n-admissible configuration. In view of Lemma 3.3, it is sufficient to verify that for the uniform $8e, P_n$-rarefication $w' = \{[b'_i; b'_i + \kappa'_i]\}_{i=1}^{2^{n+1}-3}$ of the configuration s' holds

$$\mu\Omega(w') \leq C \sum_{i=1}^{2^n - 1} \kappa'_{2i-1}$$

for an absolute constant C. For each $i \in \{1, 2, \ldots, 2^n - 2\}$ let M_i be a subset of $P_n(i)$, such that

$$\max \bigcap_{j \in M_i} [b'_{2j-1}; b'_{2j-1} + \mathrm{card} M_i \cdot \kappa'_{2j-1}] = \max \Omega(\{[b'_{2j-1}; b'_{2j-1} + \kappa'_{2j-1}]\}_{j \in P_n(i)}).$$

Evidently, whenever $b'_{2i+1} > b'_{2i} + \kappa'_{2i}$, we have $\mathrm{card} M_i > 1$ and

$$b'_{2i+1} - b'_{2i} \leq \mathrm{card} M_i \cdot \kappa'_{2i-1}.$$

By Lemma 2.4, the value $\max \mu\Omega(w')-b'_{2^{n+1}-3}$ is majorized by the total length of intervals from the configuration. So, to prove the theorem, it is enough to check that

$$\sum_{\substack{i\in\{1,2,\dots,2^n-2\}\\ b'_{2i+1}>b'_{2i}+\kappa'_{2i}}} \mathrm{card} M_i \cdot \kappa'_{2i-1} \le C_1 \sum_{i=1}^{2^n-1} \kappa'_{2i-1} \tag{22}$$

for some absolute constant C_1.

Let ρ be a number from $\{2,3,\dots,n\}$ and $R_\rho := \{i \in [1;2^n-1] : r_n(i) = \rho\}$. Next, consider a random variable ξ_ρ taking values from R_ρ with equal probability (that is, $P\{\xi_\rho = i\} = 2^{1-\rho}$, $i \in R_\rho$). Let also $\eta_m(i)$, $m = 1,2,\dots,\rho-1$, be defined on R_ρ by the formula

$$\eta_m(i) = \begin{cases} 1, & m = r_n(j) \text{ for some } j \in P_n(i),\\ 0, & else. \end{cases}$$

It is not difficult to see that $\{\eta_m(\xi_\rho)\}_{m=1}^{\rho-1}$ considered as random variables are independent, and $P\{\eta_m(\xi_\rho) = 1\} = P\{\eta_m(\xi_\rho) = 0\} = \frac{1}{2}$, $m = 1,2,\dots,\rho-1$. Indeed, for any binary sequence $\{\alpha_m\}_{m=1}^{\rho-1}$, $\alpha_m \in \{0;1\}$, $m = 1,2,\dots,\rho-1$, there exists the single $i \in R_\rho$ such that $\eta_m(i) = \alpha_m$, $m = 1,2,\dots,\rho-1$.

For any fixed $k > 1$, we are going to estimate probability of the event

$$\xi_\rho \le 2^n - 2,\ b'_{2\xi_\rho+1} > b'_{2\xi_\rho} + \kappa'_{2\xi_\rho},\ \mathrm{card} M_{\xi_\rho} = k. \tag{23}$$

Suppose, (23) is fulfilled for $\xi_\rho = i$, $i \in R_\rho$. For any number $j \in M_i$ holds $\mathrm{card} M_i \cdot \kappa'_{2j-1} > b'_{2i-1} - b'_{2j-1}$. On the other hand, if $j \ne i$ and $r'_n(j)$ is defined by

$$r'_n(j) := \min\{r > 0 : r = r_n(h) \text{ for some } h \in P_n(i) \cap \{j+1, j+2, \dots, i\}\}$$

then the set $\{j+1, j+2\dots, i\}$ contains at least one multiple of $2^{n-r'_n(j)}$. Since $M_i \subset P_n(i)$, the rank of j must be strictly less than $r'_n(j)$, so, j is also a multiple of $2^{n-r'_n(j)}$. Thus, $i - j \ge 2^{n-r'_n(j)}$, and

$$\mathrm{card} M_i \cdot \kappa'_{2j-1} > b'_{2i-1} - b'_{2j-1} \ge 2^{n-r'_n(j)} \frac{8e}{2^n-2} \sum_{q=1}^{2^n-1} \kappa'_{2q-1}.$$

The last inequality can we rewritten as follows:

$$\frac{2^{r_n(j)-1} d_{r_n(j)} k}{e \sum_{h=1}^{n} 2^{h-1} d_h} > \frac{4}{2^n-2} 2^{n-r'_n(j)+r_n(j)} > \frac{2}{2^{r'_n(j)-r_n(j)-1}}. \tag{24}$$

Note, that the left part of (24) depends only on $r_n(j)$. For any $r \in [1;\rho-1]$, let l_r be the smallest non-negative integer number, such that

$$\frac{2^{r-1} d_r k}{e \sum_{h=1}^{n} 2^{h-1} d_h} > 2^{1-l_r}.$$

Then, summing by r, we get

$$\sum_{r=1}^{\rho-1} 2^{-l_r} < \frac{k}{2e} \le \frac{k-1}{e}.$$

By definition of r'_n, the set M_i does not contain indexes with ranks in the interval $(r_n(j); r'_n(j))$ for any $j \in M_i \backslash \{i\}$. Taking into account that $l_{r_n(j)} \le r'_n(j) - r_n(j) - 1$, we get $\eta_{r_n(j)+\nu}(i) = 0,\ \nu = 1, 2, \ldots, l_{r_n(j)}$. Evidently, $\eta_{r_n(j)} = 1$, so $G_{\rho-1, r_n(j)}(\{\eta_m(i)\}_{m=1}^{\rho-1}) = 1$, with G defined as in Lemma 3.4 and $l_r,\ r = 1, 2, \ldots, \rho - 1$, determined above. Thus, conditions (23) imply

$$\sum_{r=1}^{\rho-1} G_{\rho-1,r}(\{\eta_m(i)\}_{m=1}^{\rho-1}) \ge \mathrm{card}(M_i \setminus \{i\}) = k - 1.$$

It follows that

$$P\{\xi_\rho \text{ satisfies (23)}\} \le P\{\sum_{r=1}^{\rho-1} G_{\rho-1,r}(\{\eta_m(\xi_\rho)\}_{m=1}^{\rho-1}) \ge k - 1\} \le 2^{1-k}.$$

Summing by k (recall that $k > 1$), we get

$$\sum_{\substack{i \in R_\rho \backslash \{2^n - 1\} \\ b'_{2i+1} > b'_{2i} + \kappa'_{2i}}} \mathrm{card} M_p \cdot \kappa'_{2i-1} < 2^{\rho-1} \sum_{k=2}^{\infty} k 2^{1-k} 2 d_\rho < 2^{\rho+2} d_\rho. \tag{25}$$

Note that $r_n(i) = 1$ only if $i = 2^{n-1}$, and $\mathrm{card} P_n(2^{n-1}) = 1$. It means that with necessity $b'_{2^n+1} = b'_{2^n} + \kappa'_{2^n}$. Finally, applying (25), we get

$$\sum_{\substack{i \in \{1,2,\ldots,2^n-2\} \\ b'_{2i+1} > b'_{2i} + \kappa'_{2i}}} \mathrm{card} M_i \cdot \kappa'_{2i-1} = \sum_{\rho=2}^{n} \sum_{\substack{i \in R_\rho \backslash \{2^n - 1\} \\ b'_{2i+1} > b'_{2i} + \kappa'_{2i}}} \mathrm{card} M_i \cdot \kappa'_{2p-1} < 4 \sum_{i=1}^{2^n-1} \kappa'_{2i-1}.$$

■

Finally, we consider a question whether a stronger result for generalized Cantor sets than Corollary 3.2 can be obtained. We can prove the following

PROPOSITION 3.5. *Whenever $\phi : \mathbb{N} \to \mathbb{N}$ is an increasing function with $\lim\limits_{n\to\infty} \frac{\phi(n)}{n} = \infty$, there exists a generalized Cantor set $\mathcal{C}_\phi$, such that for all $x \in \mathcal{C}_\phi$*

$$\lim_{n\to\infty} \phi(n) \left(\frac{\mu I_n}{s_n(x)} \right)^* = \infty, \tag{26}$$

where $\{I_i\}$ are intervals from the definition of $\mathcal{C}_\phi$, $s_i(x)$ is the distance from $x \in C_\phi$ to I_i and $\left(\frac{\mu I_n}{s_n(x)}\right)^*$ is the n-th element of the non-increasing rearrangement of $\left\{\frac{\mu I_i}{s_i(x)}\right\}_{i=1}^{\infty}$.

Proof. First, note that we can find positive integer numbers k_m, $k_{m+1} > k_m + 1$, $m \ge 0$, such that

$$(k_{m+1} - k_m) \uparrow \infty \tag{27}$$

and

$$\frac{\phi(i)}{i} \ge m 2^{m+2} \text{ for all } m \ge 2 \text{ and } i \ge 2^{k_{m-1} - k_{m-2} - 1}. \tag{28}$$

Next, let $\{d_i\}_{i=1}^{\infty}$ be a sequence of positive real numbers such that $d_{k_j} = 2^{-j-k_j}$, $j \ge 0$, $\sum\limits_{i=1}^{\infty} 2^{i-1} d_i < 1$ and

$$\max\{d_r : r \in [k_{m-1} + 1; k_m - 1]\} \le 2^{1-k_m},\ m \ge 2.$$

We will construct the generalized Cantor set $\mathcal{C}_\phi$ according to the algorithm described at the beginning of the section, with initial interval $[0;1]$ and d_i determined above. The set $\mathcal{C}_\phi$ shall be "homogeneous" in the sense that for each $k \in \mathbb{N}$ the neighbor intervals from $\{I_i\}_{i=1}^{2^k-1}$ are equally distanced from each other and that for the distance h_k we have:

$$h_k = \inf \bigcup_{1\le i<2^k} I_i = 1 - \sup \bigcup_{1\le i<2^k} I_i$$

(in other words, the distance between the leftmost interval and the point "0" equals the distance between the rightmost interval and "1", as well as the distance between any two neighboring intervals). Next, fix $x \in \mathcal{C}_\phi$ and $m \ge 2$. The intervals I_i with ranks not exceeding k_{m-1} split $[0;1]$ into $2^{k_{m-1}}$ closed segments K_i and, with necessity, $x \in K_j$ for some $j \in [1; 2^{k_{m-1}}]$. It is easy to see that K_j comprises $2^{k_m-k_{m-1}-1}$ intervals of rank k_m. Besides, for any two neighbor intervals I_h and I_l $(l > h)$ of rank k_m from the segment K_j,

$$\inf I_l - \inf I_h < \max\{d_r : r \in [k_{m-1}+1; k_m - 1]\} + 2^{1-k_m+k_{m-1}}\mu K_j \le \\ 2^{1-k_m} + 2^{1-k_m+k_{m-1}}\mu K_j.$$

It is also evident that the leftmost interval of rank k_m from K_j is distanced from $\inf K_j$ by not farther than $2^{-k_m+k_{m-1}}\mu K_j$; the analogous estimation holds for the rightmost interval and $\sup K_j$. Thus, if the set $M_{m,x}$ is defined by

$$M_{m,x} := \{i : I_i \text{ has rank } k_m \text{ and } I_i \subset K_j\}$$

then for any $h \in [1; 2^{k_m-k_{m-1}-1}]$

$$\operatorname{card}\{i \in M_{m,x} : s_i(x) \le h(2^{1-k_m} + 2^{1-k_m+k_{m-1}}\mu K_j)\} \ge h, \tag{29}$$

The formula (29) implies that for all $i \le 2^{k_m-k_{m-1}-1}$ we have

$$\left(\frac{\mu I_i}{s_i(x)}\right)^* \ge \frac{d_{k_m}}{i(2^{1-k_m} + 2^{1-k_m+k_{m-1}}\mu K_j)} \ge \frac{2^{-m-2}}{i}.$$

Taking into account (28), we get

$$\phi(i)\left(\frac{\mu I_i}{s_i(x)}\right)^* \ge m,\ i \in [2^{k_{m-1}-k_{m-2}-1}; 2^{k_m-k_{m-1}-1}].$$

In view of (27), we come to (26). ∎

Remark. In particular, from Proposition 3.5 one can deduce that the series (2) with $\lambda = 0$ diverges for all x for some generalized Cantor sets. For instance, we can take $\mathcal{C}_\phi$ with $\phi(n) = n\log(1+n)$.

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