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APPROXIMATION OF FUNCTIONS FROM $L^P(\omega)_\beta$ BY GENERAL LINEAR OPERATORS OF THEIR FOURIER SERIES

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Abstract. We show the general and precise conditions on the functions and modulus of continuity as well as on the entries of matrices generated the summability means and give the rates of approximation of functions from the generalized integral Lipschitz classes by double matrix means of their Fourier series. Consequently, we give some results on norm approximation. Thus we essentially extend and improve our earlier results [Acta et Commentationes Universitatis Tartuensis de Mathematica, Vol. 13 (2009), 11-24] and the result of S. Lal [Applied Mathematics and Computation 209 (2009) 346–350].

1. Introduction. Let L^p $(1 \leq p < \infty)$ $[p = \infty]$ be the class of all 2π–periodic real–valued functions (integrable in the Lebesgue sense with p–th power)[essentially bounded] over $Q = [-\pi, \pi]$ with the norm

$$\|f\| := \|f(\cdot)\|_{L^p} = \begin{cases} \left(\int_Q \mid f(t) \mid^p dt\right)^{1/p} & \text{when } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{t \in Q} \mid f(t) \mid & \text{when } p = \infty \end{cases} \tag{1.1}$$

and consider the trigonometric Fourier series

$$Sf(x) := \frac{a_o(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x) \tag{1.2}$$

with the partial sums $S_k f$.

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Let $A := (a_{n,k})$ and $B := (b_{n,k})$ be infinite lower triangular matrices of real numbers such that

$$\begin{aligned} a_{n,k} &\geq 0 \text{ and } b_{n,k} \geq 0 \text{ when } k = 0,1,2,...n, \qquad (1.3)\\ a_{n,k} &= 0 \text{ and } b_{n,k} = 0 \text{ when } k > n\ , \end{aligned}$$

$$\sum_{k=0}^{n} a_{n,k} = 1 \text{ and } \sum_{k=0}^{n} b_{n,k} = 1, \text{ where } n = 0,1,2,...\ , \qquad (1.4)$$

and let, for $m = 0,1,2,..,n$,

$$\begin{aligned} A_{n,m} &= \sum_{k=0}^{m} a_{n,k} \text{ and } \overline{A}_{n,m} = \sum_{k=m}^{n} a_{n,k} \qquad (1.5)\\ B_{n,m} &= \sum_{k=0}^{m} b_{n,k} \text{ and } \overline{B}_{n,m} = \sum_{k=m}^{n} b_{n,k}. \end{aligned}$$

Let the $AB-$transformation of $(S_k f)$ be given by

$$T_{n,A,B}f(x) := \sum_{r=0}^{n}\sum_{k=0}^{r} b_{n,r}a_{r,k}S_k f(x) \quad (\ n = 0,1,2,...)\,. \qquad (1.6)$$

As a measure of approximation by the above quantity we use the generalized modulus of continuity of f in the space L^p defined for $\beta \geq 0$ by the formula

$$\omega_\beta f(\delta)_{L^p} := \sup_{0\leq|t|\leq\delta}\left\{\left|\sin\frac{t}{2}\right|^{\beta}\|\varphi_{\cdot}(t)\|_{L^p}\right\}, \qquad (1.7)$$

where

$$\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)\,.$$

It is clear that for $\beta > \alpha \geq 0$

$$\omega_\beta f(\delta)_{L^p} \leq \omega_\alpha f(\delta)_{L^p}\ ,$$

and it is easily seen that $\omega_0 f(\cdot)_{L^p} = \omega f(\cdot)_{L^p}$ is the classical modulus of continuity.

The deviation $T_{n,A,B}f - f$ with the lower triangular infinite matrix B, defined by $b_{n,r} = \frac{1}{n+1}$ when $r = 0,1,2,...n$ and $b_{n,r} = 0$ when $r > n$, and with the lower triangular infinite matrix A, defined by $a_{r,k} = p_{r-k}/\sum_{\nu=0}^{r} p_\nu$ when $k = 0,1,2,...r$ and $a_{r,k} = 0$ when $k > r$, was estimated by S. Lal [1, Theorem 2] as follows:

Theorem A. *If*

$$\begin{aligned} f &\in L_\beta^p(\omega)\\ &= \left\{ f \in L^p : \ \omega f(\delta)_{L_\beta^p} := \sup_{0\leq|t|\leq\delta}\left\{\int_0^\pi |\varphi_x(t)|^p\left|\sin\frac{x}{2}\right|^{\beta p}dx\right\}^{\frac{1}{p}} \leq \omega(\delta)\right\}, \end{aligned}$$

where ω *is such that*

$$\frac{\omega(t)}{t} \text{ is a decreasing function of } t,$$

$$\left\{\int_{0}^{\pi/(n+1)}\left(\frac{t\left|\varphi_{x}(t)\right|}{\omega(t)}\right)^{p}\sin^{\beta p}t dt\right\}^{1/p}=O\left((n+1)^{-1}\right), \tag{1.8}$$

and

$$\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\gamma}\left|\varphi_{x}(t)\right|}{\omega(t)}\right)^{p}\sin^{\beta p}t dt\right\}^{1/p}=O\left((n+1)^{\gamma}\right)\quad (0<\gamma<\frac{1}{p}), \tag{1.9}$$

then

$$\left\|\frac{1}{n+1}\sum_{\nu=0}^{n}\frac{1}{P_{\nu}}\sum_{k=0}^{\nu}p_{\nu-k}S_{k}f-f\right\|_{L^{p}}=O\left((n+1)^{\beta+\frac{1}{p}}\omega\left(\frac{1}{n+1}\right)\right), \tag{1.10}$$

where $P_{n}=\sum_{\nu=0}^{n}p_{\nu}$ *with nonnegative and nonincreasing sequence* (p_{ν})*.*

Since the condition (1.8) used in the estimate of the integral $\int_{0}^{\frac{\pi}{n+1}}$ (from the proof of Theorem 2 [1] of S. Lal) leads us to the divergent integral of the form $\int_{0}^{\frac{\pi}{n+1}}t^{\frac{-(1+\beta)}{1-1/p}}dt$ under the assumption $\beta\geq 0$, therefore, instead of this condition, we shall take the following one:

$$\left\{\int_{0}^{\pi/(n+1)}\left(\frac{\left|\varphi_{x}(t)\right|}{\omega(t)}\right)^{p}\sin^{\beta p}t dt\right\}^{1/p}=O_{x}\left((n+1)^{-1/p}\right). \tag{1.11}$$

In the paper [1, (Proof of Theorem 2)] $\sin\frac{t}{2}$ should be used instead of $\sin t$.

In our theorems we will consider the pointwise deviation

$$T_{n,A,B}f(x)-f(x)$$

with the mean $T_{n,A,B}f$ introduced at the beginning. We will formulate general and precise conditions on the functions and the modulus of continuity as well as on the entries of the matrices A and B and give the rates of approximation of functions from the generalized integral Lipschitz classes by our double matrix means of their Fourier series. Consequently, we give some results on norm approximation. Thus we essentially extend and improve our earlier results (see [2]) and the result of S. Lal [1].

We shall write $I_{1}\ll I_{2}$ if there exists a positive constant K, sometimes depending on some parameters, such that $I_{1}\leq KI_{2}$.

2. Statement of the results. Let us consider a function ω of modulus of continuity type on the interval $[0,2\pi]$, i.e. a nondecreasing continuous function having the following properties: $\omega(0)=0$, $\omega(\delta_{1}+\delta_{2})\leq\omega(\delta_{1})+\omega(\delta_{2})$ for any $0\leq\delta_{1}\leq\delta_{2}\leq\delta_{1}+\delta_{2}\leq 2\pi$. It is easy to conclude that the function $\delta^{-1}\omega(\delta)$ is a quasi nonincreasing function of δ. Let, for such ω,

$$L^{p}(\omega)_{\beta}=\left\{f\in L^{p}:\omega_{\beta}f(\delta)_{L^{p}}\leq\omega(\delta)\right\}.$$

It is clear that, for $\beta\geq\alpha\geq 0$,

$$L^{p}(\omega)_{\alpha}\subset L^{p}(\omega)_{\beta}\ .$$

Now, we can formulate our main results on the degrees of pointwise summability.

THEOREM 1. *Let* $f \in L^p(\omega)_\beta$ *with* $0 \leq \beta < 1 - \frac{1}{p}$, *and let* ω *satisfy*

$$\begin{array}{c} \left\{ \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \left(\frac{|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{\frac{1}{p}} = O_x\left((n+1)^{-\frac{2}{p}} \right), \text{when } \ 1 < p < \infty, \\ ess \sup_{t \in [\frac{\pi}{n+1}, \frac{\pi}{n}]} \mid \frac{|\varphi_x(t)|}{\omega(t)} \sin^\beta \frac{t}{2} \mid = O_x(1), \ \text{when } \ p = \infty \end{array} \tag{2.1}$$

and

$$\begin{array}{c} \left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{\frac{1}{p}} = O_x\left((n+1)^{-\frac{1}{p}} \right), \ \text{when } \ 1 < p < \infty, \\ ess \sup_{t \in [0, \frac{\pi}{n+1}]} \mid \frac{|\varphi_x(t)|}{\omega(t)} \sin^\beta \frac{t}{2} \mid = O_x(1), \ \text{when } \ p = \infty. \end{array} \tag{2.2}$$

If the entries of our matrices satisfy conditions

$$b_{n,n} \ll \frac{1}{n+1} \tag{2.3}$$

and

$$|b_{n,r} a_{r,r-l} - b_{n,r+1} a_{r+1,r+1-l}| \ll \frac{b_{n,r}}{(r+1)^2} \text{ for } 0 \leq l \leq r \leq n-1, \tag{2.4}$$

then

$$\begin{aligned} \left| T_{n,A,B} f(x) - f(x) \right| &= O_x \left(\sum_{r=0}^{n} b_{n,r} \frac{1}{r+1} \sum_{s=0}^{r} (s+1)^\beta \omega\left(\frac{\pi}{s+1} \right) \right. \\ & \left. + \frac{1}{n+1} \sum_{s=0}^{n} (s+1)^\beta \omega\left(\frac{\pi}{s+1} \right) \right) \end{aligned}$$

and, in the case $0 < \beta < 1 - \frac{1}{p}$,

$$\begin{aligned} & \left| T_{n,A,B} f(x) - f(x) \right| \\ &= O_x \left((n+1)^\beta \omega\left(\frac{\pi}{n+1} \right) \left[(n+1)^{1-\beta} \sum_{s=0}^{n} b_{n,s} (s+1)^{\beta-1} \right] \right), \end{aligned}$$

for considered x.

THEOREM 2. *Let* $f \in L^p(\omega)_\beta$ *with* $0 \leq \beta < 1 - \frac{1}{p}$, *and let* ω *satisfy (2.2) and*

$$\begin{array}{c} \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{|\varphi_x(t)|}{t^\gamma \omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{\frac{1}{p}} = O_x\left((n+1)^\gamma \right), \ \text{when } \ 1 < p < \infty, \\ ess \sup_{t \in [\frac{\pi}{n+1}, \pi]} \mid \frac{|\varphi_x(t)|}{t^\gamma \omega(t)} \sin^\beta \frac{t}{2} \mid = O_x\left((n+1)^\gamma \right), \ \text{when } \ p = \infty, \end{array} \tag{2.5}$$

with a nonnegative γ *such that* $\beta - \gamma < 1 - \frac{1}{p}$. *If the entries of our matrices satisfy the conditions (2.3) and (2.4), then*

$$\begin{aligned} & \left| T_{n,A,B} f(x) - f(x) \right| \\ &= O_x \left(\left\{ (n+1)^{\gamma q} \sum_{r=0}^{n} b_{n,r} \frac{1}{r+1} \sum_{s=0}^{r} \left(\omega\left(\frac{\pi}{s+1} \right) (s+1)^{\beta-\gamma+1/p} \right)^q \right\}^{\frac{1}{q}} \right. \\ & \left. + \left\{ (n+1)^{\gamma q - 1} \sum_{s=0}^{n} \left(\omega\left(\frac{\pi}{s+1} \right) (s+1)^{\beta-\gamma+1/p} \right)^q \right\}^{\frac{1}{q}} \right) \end{aligned}$$

and, in the case $0<\beta-\gamma<1-\frac{1}{p}$,

$$\begin{aligned}
&\left|T_{n,A,B}f(x)-f(x)\right| \\
&= O_x\left((n+1)^{\beta+1/p}\,\omega\left(\frac{\pi}{n+1}\right)\left\{(n+1)^{1-(\beta-\gamma)q}\sum_{r=0}^{n}b_{n,r}(r+1)^{(\beta-\gamma)q-1}\right\}^{\frac{1}{q}}\right),
\end{aligned}$$

for considered x, *where* $q=\frac{p}{p-1}$.

If the entries of the matrix B are as in Theorem A then we can formulate the above theorem in the following simpler form.

THEOREM 3. *Let* $f\in L^{p}(\omega)_{\beta}$ *with* $0\leq\beta<1-\frac{1}{p}$, *and let* ω *satisfy (2.1) and (2.2). If the entries of the matrix* A *satisfy the condition*

$$\left|a_{r,r-l}-a_{r+1,r+1-l}\right|\ll\frac{1}{(r+1)^{2}}\ \text{for } 0\leq l\leq r, \tag{2.6}$$

then

$$\begin{aligned}
&\left|T_{n,A,\left(\frac{1}{n+1}\right)}f(x)-f(x)\right| \\
&= O_x\left(\frac{1}{n+1}\sum_{r=0}^{n}\frac{1}{r+1}\sum_{s=0}^{r}(s+1)^{\beta}\,\omega\left(\frac{\pi}{s+1}\right)\right)
\end{aligned}$$

and, for $0<\beta<1-\frac{1}{p}$,

$$\left|T_{n,A,\left(\frac{1}{n+1}\right)}f(x)-f(x)\right|=O_x\left((n+1)^{\beta}\,\omega\left(\frac{\pi}{n+1}\right)\right),$$

for considered x.

THEOREM 4. *Let* $f\in L^{p}(\omega)_{\beta}$ *with* $0\leq\beta<1-\frac{1}{p}$, *and let* ω *satisfy (2.2) and (2.5) with a nonnegative* γ *such that* $\beta-\gamma<1-\frac{1}{p}$. *If the entries of the matrix* A *satisfy condition (2.6), then*

$$\begin{aligned}
&\left|T_{n,A,\left(\frac{1}{n+1}\right)}f(x)-f(x)\right| \\
&= O_x\left(\left\{(n+1)^{\gamma q-1}\sum_{r=0}^{n}\frac{1}{r+1}\sum_{s=0}^{r}\left(\omega\left(\frac{\pi}{s+1}\right)(s+1)^{\beta-\gamma+1/p}\right)^{q}\right\}^{\frac{1}{q}}\right),
\end{aligned}$$

where $q=\frac{p}{p-1}$ *and, in the case* $0<\beta-\gamma<1-\frac{1}{p}$,

$$\left|T_{n,A,\left(\frac{1}{n+1}\right)}f(x)-f(x)\right|=O_x\left((n+1)^{\beta+1/p}\,\omega\left(\frac{\pi}{n+1}\right)\right),$$

for considered x.

COROLLARY 1. Under the assumptions of Theorem 4 on a function f, if (p_{ν}) is a nonincreasing sequence such that

$$P_{\tau}\sum_{\nu=\tau}^{n}P_{\nu}^{-1}=O(\tau)\quad\text{for any } \tau\geq 0, \tag{2.7}$$

then from Theorem 4 we obtain the corrected form of the result of S. Lal.

REMARK 1. We note that, in the proof of the mentioned theorem of S. Lal [1] the condition

$$P_\tau \sum_{\nu=\tau}^{n} P_\nu^{-1} = O\left(n+1\right) \quad \text{for any } \tau \geq 0,$$

is used, which holds for every nonnegative sequences (p_k). Instead (2.7) should be used.

Consequently, we reformulate the results on the L^p estimate of the norm of the deviation considered above.

THEOREM 5. *Let* $f \in L^p\left(\omega\right)_\beta$ *with* $0 \leq \beta < 1 - \frac{1}{p}$. *If the entries of our matrices satisfy conditions (2.3) and (2.4), then*

$$\begin{aligned} \left\| T_{n,A,B} f\left(\cdot\right) - f\left(\cdot\right) \right\|_{L^p} &= O_x \left(\sum_{r=0}^{n} b_{n,r} \frac{1}{r+1} \sum_{s=0}^{r} \left(s+1\right)^\beta \omega\left(\frac{\pi}{s+1}\right) \right. \\ &\quad \left. + \frac{1}{n+1} \sum_{s=0}^{n} \left(s+1\right)^\beta \omega\left(\frac{\pi}{s+1}\right) \right) \end{aligned}$$

and, for $0 < \beta < 1 - \frac{1}{p}$,

$$\begin{aligned} &\left\| T_{n,A,B} f\left(\cdot\right) - f\left(\cdot\right) \right\|_{L^p} \\ &= O_x \left(\left(n+1\right)^\beta \omega\left(\frac{\pi}{n+1}\right) \left[\left(n+1\right)^{1-\beta} \sum_{s=0}^{n} b_{n,s} \left(s+1\right)^{\beta-1} \right] \right). \end{aligned}$$

THEOREM 6. *Let* $f \in L^p\left(\omega\right)_\beta$ *with* $0 \leq \beta < 1 - \frac{1}{p}$. *If the entries of matrix A satisfy the condition (2.6), then*

$$\begin{aligned} &\left\| T_{n,A,\left(\frac{1}{n+1}\right)} f\left(\cdot\right) - f\left(\cdot\right) \right\|_{L^p} \\ &= O_x \left(\frac{1}{n+1} \sum_{r=0}^{n} \frac{1}{r+1} \sum_{s=0}^{r} \left(s+1\right)^\beta \omega\left(\frac{\pi}{s+1}\right) \right) \end{aligned}$$

and, in the case $0 < \beta < 1 - \frac{1}{p}$,

$$\left\| T_{n,A,\left(\frac{1}{n+1}\right)} f\left(\cdot\right) - f\left(\cdot\right) \right\|_{L^p} = O\left(\left(n+1\right)^\beta \omega\left(\frac{\pi}{n+1}\right) \right).$$

REMARK 2. In the case if $p \geq 1$ (specially if $p = 1$) we can suppose that the expression $t^{-\beta}\omega\left(t\right)$ is nondecreasing in t instead of the assumption $\beta < 1 - \frac{1}{p}$.

REMARK 3. Under the additional assumptions $\beta = 0$ and $\omega\left(t\right) = O\left(t^\alpha\right)$ $(0 < \alpha < 1)$, the degree of approximation in Theorem 3 is $O\left(n^{-\alpha}\right)$, but in Theorem 4, is $O\left(n^{\frac{1}{p}-\alpha}\right)$.

REMARK 4. If we consider the modulus of continuity $\omega f\left(\delta\right)_{L^p_\beta}$, then our theorems will be true under the assumption that $f \in L^p_\beta\left(\omega\right)$ and with the following norm

$$\|f\|_{L^p_\beta} := \|f(\cdot)\|_{L^p_\beta} = \begin{cases} \left(\int_Q \mid f(t) \mid^p \left|\sin \frac{t}{2}\right|^{\beta p} dt \right)^{1/p} & \text{when } 1 \leq p < \infty, \\ \underset{t \in Q}{ess\sup} \left\{ \mid f(t) \mid \left|\sin \frac{t}{2}\right|^\beta \right\} & \text{when } p = \infty. \end{cases}$$

3. Proofs of the results. We begin this section by some notations following A. Zygmund [3].

It is clear that

$$S_k f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, D_k(t)\, dt$$

and

$$T_{n,A,B} f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t) \sum_{r=0}^{n}\sum_{k=0}^{r} b_{n,r} a_{r,k} D_k(t)\, dt,$$

where

$$D_k(t) = \frac{1}{2} + \sum_{\nu=1}^{k} \cos \nu t = \begin{cases} \frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}} & \text{for} \quad k \neq 2\pi r, \quad r = 0,1,2,... \\ k+\frac{1}{2} & \text{otherwise} \end{cases}$$

and

$$|D_k(t)| \leq \begin{cases} \frac{\pi}{|t|} & \text{when } 0 < |t| \leq \pi, \\ k+1 & \text{when } t \in (-\infty, +\infty). \end{cases}$$

Hence

$$T_{n,A,B} f(x) - f(x) = \frac{1}{\pi}\int_{0}^{\pi} \varphi_x(t) \sum_{r=0}^{n}\sum_{k=0}^{r} b_{n,r} a_{r,k} D_k(t)\, dt.$$

We will prove our results for $1 < p < \infty$ only. If $p = \infty$ we have to use the generalized Hölder inequality instead the classical one.

3.1. Proof of Theorem 1. Let

$$\begin{aligned} T_{n,A,B}(x) - f(x) &= \frac{1}{\pi}\left(\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}\right) \varphi_x(t) \sum_{r=0}^{n}\sum_{k=0}^{r} b_{n,r} a_{r,k} D_k(t)\, dt \\ &= \int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi} \end{aligned}$$

and

$$|T_{n,A,B} f(x) - f(x)| \leq \left|\int_0^{\frac{\pi}{n+1}}\right| + \left|\int_{\frac{\pi}{n+1}}^{\pi}\right|.$$

By the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, and (2.2), for $0 \leq \beta < 1 - \frac{1}{p}$,

$$\begin{aligned} \left|\int_0^{\frac{\pi}{n+1}}\right| &\leq \frac{(n+1)}{\pi}\int_0^{\frac{\pi}{n+1}} |\varphi_x(t)|\, dt \\ &\leq \frac{(n+1)}{\pi}\left\{\int_0^{\frac{\pi}{n+1}} \left[\frac{|\varphi_x(t)|}{\omega(t)} \sin^\beta \frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}} \left\{\int_0^{\frac{\pi}{n+1}} \left[\frac{\omega(t)}{\sin^\beta \frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&\ll (n+1)^{1-\frac{1}{p}}\,\omega\left(\frac{\pi}{n+1}\right)\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{1}{t^{\beta}}\right]^q dt\right\}^{\frac{1}{q}} \\
&\ll (n+1)^{\beta}\,\omega\left(\frac{\pi}{n+1}\right) \ll \frac{1}{n+1}\sum_{s=1}^{n}(s+1)^{\beta}\,\omega\left(\frac{\pi}{n+1}\right) \\
&\le \frac{1}{n+1}\sum_{s=1}^{n}(s+1)^{\beta}\,\omega\left(\frac{\pi}{s+1}\right)
\end{aligned}
$$

and, since $\sin\frac{t}{2}\geq\frac{t}{\pi}$ or $\left|\sin\left(2k+1\right)\frac{t}{2}\right|\leq\left(2k+1\right)\sin\frac{t}{2}$ for $t\in\left[0,\pi\right]$, we have

$$
\left|\int_{\frac{\pi}{n+1}}^{\pi}\right| \le \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\left|\varphi_x(t)\sum_{r=0}^{n}\sum_{k=0}^{r}b_{n,r}a_{r,k}\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right|dt
$$

$$
\begin{aligned}
\ll\ & \int_{\frac{\pi}{n+1}}^{\pi}\left|\varphi_x(t)\right|\left|\sum_{r=0}^{\tau-1}\sum_{k=0}^{r}(k+1)\,b_{n,r}a_{r,k}\right|dt \\
&+\int_{\frac{\pi}{n+1}}^{\pi}\frac{\left|\varphi_x(t)\right|}{t}\left|\sum_{r=\tau}^{n}\sum_{k=0}^{\tau-1}b_{n,r}a_{r,r-k}\sin\left(r-k+\frac{1}{2}\right)t\right|dt \\
&+\int_{\frac{\pi}{n+1}}^{\pi}\frac{\left|\varphi_x(t)\right|}{t}\left|\sum_{r=\tau}^{n}\sum_{k=\tau}^{r}b_{n,r}a_{r,r-k}\sin\left(r-k+\frac{1}{2}\right)t\right|dt \\
&= I_1+I_2+I_3,
\end{aligned}
$$

where $\tau=\left[\frac{\pi}{t}\right]$ for $t\in(0,\pi]$.

Now we shall estimate the integrals of type I. So, using the Hölder inequality, by the assumption (2.1)

$$
I_1 \le \int_{\frac{\pi}{n+1}}^{\pi}\left|\varphi_x(t)\right|\sum_{r=0}^{\tau-1}(r+1)\,b_{n,r}dt
$$

$$
=\sum_{s=1}^{n}\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\left|\varphi_x(t)\right|\sum_{r=0}^{\tau-1}(r+1)\,b_{n,r}dt \ll \sum_{s=1}^{n}\sum_{r=0}^{s}(r+1)\,b_{n,r}\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\left|\varphi_x(t)\right|dt
$$

$$
\le\sum_{s=1}^{n}\sum_{r=0}^{s}(r+1)\,b_{n,r}\left\{\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\left[\frac{\left|\varphi_x(t)\right|}{\omega(t)}\sin^{\beta}\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\left[\frac{\omega(t)}{\sin^{\beta}\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}}
$$

$$
\begin{aligned}
&\ll \sum_{s=0}^{n}\sum_{r=0}^{s}(r+1)\,b_{n,r}\,(s+1)^{-\frac{2}{p}}\,\omega\left(\frac{\pi}{s+1}\right)(s+1)^{\beta-2/q} \\
&\le \sum_{s=0}^{n}\sum_{r=0}^{s}(r+1)\,b_{n,r}\omega\left(\frac{\pi}{s+1}\right)(s+1)^{\beta-2} \\
&= \sum_{r=0}^{n}(r+1)\,b_{n,r}\sum_{s=r}^{n}\omega\left(\frac{\pi}{s+1}\right)(s+1)^{\beta-2}
\end{aligned}
$$

$$\begin{aligned}
&\le \sum_{r=0}^{n}(r+1)\,b_{n,r}\omega\left(\frac{\pi}{r+1}\right)(r+1)^{\beta-1}\\
&= \sum_{r=0}^{n} b_{n,r}\omega\left(\frac{\pi}{r+1}\right)(r+1)^{\beta}\\
&\ll \sum_{r=0}^{n} b_{n,r}\frac{1}{r+1}\sum_{s=0}^{r}(s+1)^{\beta}\,\omega\left(\frac{\pi}{s+1}\right).
\end{aligned}$$

Since (2.3) and (2.4) we have

$$\begin{aligned}
&I_2\\
&= \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t}\left|\sum_{k=0}^{\tau-1}\left[\sum_{r=\tau}^{n-1}\left(b_{n,r}a_{r,r-k}-b_{n,r+1}a_{r+1,r+1-k}\right)\sum_{l=\tau}^{r}\sin\left(l-k+\frac{1}{2}\right)t\right.\right.\\
&\quad\left.\left.+\,b_{n,n}a_{n,n-k}\sum_{l=\tau}^{n}\sin\left(l-k+\frac{1}{2}\right)t\right]\right|dt
\end{aligned}$$

$$\begin{aligned}
&\ll \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}\sum_{k=0}^{\tau}\left[\sum_{r=\tau}^{n-1}\frac{b_{n,r}}{(r+1)^2}+b_{n,n}a_{n,n-k}\right]dt\\
&\le \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}\left[(\tau+1)\sum_{r=\tau}^{n-1}\frac{b_{n,r}}{(r+1)^2}+b_{n,n}\sum_{k=0}^{\tau}a_{n,n-k}\right]dt\\
&\le \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}\left[(\tau+1)\sum_{r=\tau}^{n-1}\frac{b_{n,r}}{(r+1)^2}+b_{n,n}\right]dt
\end{aligned}$$

$$\begin{aligned}
&\le \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}(\tau+1)\sum_{r=\tau}^{n-1}\frac{b_{n,r}}{(r+1)^2}dt+b_{n,n}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}dt\\
&\le \sum_{s=1}^{n}\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\frac{|\varphi_x(t)|}{t^2}(\tau+1)\sum_{r=\tau}^{n-1}\frac{b_{n,r}}{(r+1)^2}dt+\frac{1}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}dt\\
&\le \sum_{s=1}^{n}\left((s+1)\sum_{r=s}^{n}\frac{b_{n,r}}{(r+1)^2}+\frac{1}{n+1}\right)\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\frac{|\varphi_x(t)|}{t^2}dt
\end{aligned}$$

$$\begin{aligned}
&\le \sum_{s=1}^{n}\left((s+1)\sum_{r=s}^{n}\frac{b_{n,r}}{(r+1)^2}+\frac{1}{n+1}\right)\\
&\quad\cdot\left\{\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\left[\frac{|\varphi_x(t)|}{\omega(t)}\sin^{\beta}\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\left[\frac{\omega(t)}{t^2\sin^{\beta}\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}}\\
&\ll \sum_{s=1}^{n}\left((s+1)\sum_{r=s}^{n}\frac{b_{n,r}}{(r+1)^2}+\frac{1}{n+1}\right)(n+1)^{-\frac{2}{p}}\,\omega\left(\frac{\pi}{s}\right)\left\{\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\left[\frac{1}{t^{\beta+2}}\right]^q dt\right\}^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\ll \sum_{s=0}^{n}\left((s+1)\sum_{r=s}^{n}\frac{b_{n,r}}{(r+1)^{2}}+\frac{1}{n+1}\right)(s+1)^{\beta}\,\omega\left(\frac{\pi}{s+1}\right)\\
&\le \sum_{r=0}^{n}\frac{b_{n,r}}{(r+1)^{2}}\sum_{s=0}^{r}(s+1)^{\beta+1}\,\omega\left(\frac{\pi}{s+1}\right)+\frac{1}{n+1}\sum_{s=0}^{n}(s+1)^{\beta}\,\omega\left(\frac{\pi}{s+1}\right)\\
&\le \sum_{r=0}^{n}b_{n,r}\frac{1}{r+1}\sum_{s=0}^{r}(s+1)^{\beta}\,\omega\left(\frac{\pi}{s+1}\right)\\
&\quad+\frac{1}{n+1}\sum_{s=0}^{n}(s+1)^{\beta}\,\omega\left(\frac{\pi}{s+1}\right)\\
&\le \sum_{r=0}^{n}b_{n,r}\frac{1}{r+1}\sum_{s=0}^{r}(s+1)^{\beta}\,\omega\left(\frac{\pi}{s+1}\right)+\frac{1}{n+1}\sum_{s=0}^{n}(s+1)^{\beta}\,\omega\left(\frac{\pi}{s+1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&I_3\\
&= \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t}\left|\sum_{r=\tau}^{n}\sum_{k=\tau}^{r}b_{n,r}a_{r,r-k}\sin\left(r-k+\frac{1}{2}\right)t\right|dt\\
&= \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t}\left|\sum_{k=\tau}^{n}\sum_{r=k}^{n}b_{n,r}a_{r,r-k}\sin\left(r-k+\frac{1}{2}\right)t\right|dt\\
&= \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t}\left|\sum_{k=\tau}^{n}\left[\sum_{r=k}^{n-1}(b_{n,r}a_{r,r-k}-b_{n,r+1}a_{r+1,r+1-k})\sum_{l=k}^{r}\sin\left(l-k+\frac{1}{2}\right)t\right.\right.\\
&\quad\left.\left.+\,b_{n,n}a_{n,n-k}\sum_{l=k}^{n}\sin\left(l-k+\frac{1}{2}\right)t\right]\right|dt\\
&\ll \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}\sum_{k=\tau}^{n}\left[\sum_{r=k}^{n-1}|b_{n,r}a_{r,r-k}-b_{n,r+1}a_{r+1,r+1-k}|+b_{n,n}a_{n,n-k}\right]dt\\
&\le \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}\left[\sum_{r=\tau}^{n}\sum_{k=\tau}^{r}|b_{n,r}a_{r,r-k}-b_{n,r+1}a_{r+1,r+1-k}|+b_{n,n}\sum_{k=\tau}^{n}a_{n,n-k}\right]dt\\
&\ll \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}\left[\sum_{r=\tau}^{n}\sum_{k=\tau}^{r}\frac{b_{n,r}}{(r+1)^{2}}+\frac{1}{n+1}\right]dt\\
&\le \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}\left[\sum_{r=\tau}^{n}\frac{b_{n,r}}{r+1}+\frac{1}{n+1}\right]dt.
\end{aligned}
$$

Further, the same calculation, as that in the estimate of I_2 , gives the inequality

$$
\begin{aligned}
I_3 &\ll \sum_{s=1}^{n}\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\frac{|\varphi_x(t)|}{t^2}\left[\sum_{r=\tau}^{n}\frac{b_{n,r}}{r+1}+\frac{1}{n+1}\right]dt\\
&= \sum_{s=1}^{n}\left[\sum_{r=s}^{n}\frac{b_{n,r}}{r+1}+\frac{1}{n+1}\right]\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\frac{|\varphi_x(t)|}{t^2}dt
\end{aligned}
$$

$$\ll \sum_{r=0}^{n} b_{n,r}\frac{1}{r+1}\sum_{s=0}^{r}(s+1)^\beta\,\omega\left(\frac{\pi}{s+1}\right)+\frac{1}{n+1}\sum_{s=0}^{n}(s+1)^\beta\,\omega\left(\frac{\pi}{s+1}\right).$$

If $\beta>0$, then

$$\begin{aligned}
&|T_{n,A,B}f(x)-f(x)|\\
\ll\ & (n+1)\,\omega\left(\frac{\pi}{n+1}\right)\sum_{s=0}^{n}\left(b_{n,s}+\frac{1}{n+1}\right)(s+1)^{\beta-1}\\
\le\ & (n+1)\,\omega\left(\frac{\pi}{n+1}\right)\left[\sum_{s=0}^{n}b_{n,s}(s+1)^{\beta-1}+\frac{1}{n+1}\sum_{s=0}^{n}(s+1)^{\beta-1}\right]\\
=\ & (n+1)^\beta\,\omega\left(\frac{\pi}{n+1}\right)\left[(n+1)^{1-\beta}\sum_{s=0}^{n}b_{n,s}(s+1)^{\beta-1}+1\right]\\
\le\ & (n+1)^\beta\,\omega\left(\frac{\pi}{n+1}\right)\left[(n+1)^{1-\beta}\sum_{s=0}^{n}b_{n,s}(s+1)^{\beta-1}\right]
\end{aligned}$$

Collecting these estimates we obtain the desired result. ■

3.2. Proof of Theorem 2. With the notations of the above proof,

$$\begin{aligned}
\left|\int_0^{\frac{\pi}{n+1}}\right| \ll\ & \frac{1}{r+1}\sum_{s=1}^{r}(s+1)^\beta\,\omega\left(\frac{\pi}{s+1}\right)\\
\le\ & \left\{\frac{1}{r+1}\sum_{s=1}^{r}\left((s+1)^\beta\,\omega\left(\frac{\pi}{s+1}\right)\right)^q\right\}^{\frac{1}{q}}\\
\le\ & \left\{(r+1)^{\gamma q-1}\sum_{s=1}^{r}\left((s+1)^{\beta-\gamma}\,\omega\left(\frac{\pi}{s+1}\right)\right)^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Furthermore, using the Hölder inequality, by the condition (2.5), we obtain the next estimates

$$I_1\le\int_{\frac{\pi}{n+1}}^{\pi}|\varphi_x(t)|\sum_{r=0}^{\tau-1}(r+1)\,b_{n,r}dt$$

$$\le\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{|\varphi_x(t)|}{t^\gamma\omega(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{t^\gamma\omega(t)}{\sin^\beta\frac{t}{2}}\sum_{r=0}^{\tau-1}(r+1)\,b_{n,r}\right]^q dt\right\}^{\frac{1}{q}}$$

$$\begin{aligned}
\ll\ & (n+1)^\gamma\left\{\sum_{s=1}^{n}\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\left[\frac{t^\gamma\omega(t)}{\sin^\beta\frac{t}{2}}\sum_{r=0}^{\tau-1}(r+1)\,b_{n,r}\right]^q dt\right\}^{\frac{1}{q}}\\
\ll\ & (n+1)^\gamma\left\{\sum_{s=1}^{n}\left(\sum_{r=0}^{s}(r+1)\,b_{n,r}\omega\left(\frac{\pi}{s+1}\right)(s+1)^{\beta-\gamma}\right)^q\left(\frac{\pi}{s}-\frac{\pi}{s+1}\right)\right\}^{\frac{1}{q}}\\
\ll\ & (n+1)^\gamma\left\{\sum_{s=0}^{n}\left(\omega\left(\frac{\pi}{s+1}\right)\right)^q(s+1)^{(\beta-\gamma)q-2}\left(\sum_{r=0}^{s}(r+1)\,b_{n,r}\right)^q\right\}^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\ll (n+1)^{\gamma}\left\{\sum_{s=0}^{n}\left(\omega\left(\frac{\pi}{s+1}\right)\right)^{q}(s+1)^{(\beta-\gamma)q-2}\sum_{r=0}^{s}(r+1)^{q}\,b_{n,r}\right\}^{\frac{1}{q}}\\
&\ll (n+1)^{\gamma}\left\{\sum_{r=0}^{n}(r+1)^{q}\,b_{n,r}\sum_{s=r}^{n}\left(\omega\left(\frac{\pi}{s+1}\right)\right)^{q}(s+1)^{(\beta-\gamma)q-2}\right\}^{\frac{1}{q}}\\
&\le (n+1)^{\gamma}\left\{\sum_{r=0}^{n}b_{n,r}\left(\omega\left(\frac{\pi}{r+1}\right)\right)^{q}(r+1)^{(\beta-\gamma)q+q-1}\right\}^{\frac{1}{q}}\\
&\le (n+1)^{\gamma}\left\{\sum_{r=0}^{n}b_{n,r}\left(\omega\left(\frac{\pi}{r+1}\right)(r+1)^{\beta-\gamma+1/p}\right)^{q}\right\}^{\frac{1}{q}}
\end{aligned}
$$

$$
\le (n+1)^{\gamma}\left\{\sum_{r=0}^{n}b_{n,r}\frac{1}{r+1}\sum_{s=0}^{r}\left(\omega\left(\frac{\pi}{s+1}\right)(s+1)^{\beta-\gamma+1/p}\right)^{q}\right\}^{\frac{1}{q}}
$$

and

$$
I_2 \ll \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}\tau\sum_{r=\tau}^{n-1}\frac{b_{n,r}}{(r+1)^2}dt + b_{n,n}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}dt
$$

$$
\begin{aligned}
&\le \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}\sum_{r=\tau}^{n-1}\frac{b_{n,r}}{r+1}dt+\frac{1}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t^2}dt\\
&\le \left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{|\varphi_x(t)|}{t^{\gamma}\omega(t)}\sin^{\beta}\frac{t}{2}\right]^{p}dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{t^{\gamma}\omega(t)}{t^2\sin^{\beta}\frac{t}{2}}\sum_{r=\tau}^{n-1}\frac{b_{n,r}}{r+1}\right]^{q}dt\right\}^{\frac{1}{q}}\\
&\quad+\frac{1}{n+1}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{|\varphi_x(t)|}{t^{\gamma}\omega(t)}\sin^{\beta}\frac{t}{2}\right]^{p}dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{t^{\gamma}\omega(t)}{t^2\sin^{\beta}\frac{t}{2}}\right]^{q}dt\right\}^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\ll (n+1)^{\gamma}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{t^{\gamma}\omega(t)}{t^2\sin^{\beta}\frac{t}{2}}\sum_{r=\tau}^{n-1}\frac{b_{n,r}}{r+1}\right]^{q}dt\right\}^{\frac{1}{q}}\\
&\quad+\frac{(n+1)^{\gamma}}{n+1}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{t^{\gamma}\omega(t)}{t^2\sin^{\beta}\frac{t}{2}}\right]^{q}dt\right\}^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&= (n+1)^{\gamma}\left\{\sum_{s=1}^{n}\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\left[\frac{t^{\gamma}\omega(t)}{t^2\sin^{\beta}\frac{t}{2}}\sum_{r=\tau}^{n-1}\frac{b_{n,r}}{r+1}\right]^{q}dt\right\}^{\frac{1}{q}}\\
&\quad+\frac{(n+1)^{\gamma}}{n+1}\left\{\sum_{s=1}^{n}\int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}}\left[\frac{t^{\gamma}\omega(t)}{t^2\sin^{\beta}\frac{t}{2}}\right]^{q}dt\right\}^{\frac{1}{q}}
\end{aligned}
$$

$$\leq \quad (n+1)^\gamma \left\{ \sum_{s=1}^{n} \left(\sum_{r=s}^{n-1} \frac{b_{n,r}}{r+1} \right)^q \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \left[\frac{t^\gamma \omega(t)}{t^2 \sin^\beta \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}}$$

$$+ \frac{(n+1)^\gamma}{n+1} \left\{ \sum_{s=1}^{n} \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \left[\frac{t^\gamma \omega(t)}{t^2 \sin^\beta \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}}$$

$$\leq \quad (n+1)^\gamma \left\{ \sum_{s=1}^{n} \left(\sum_{r=s}^{n-1} \frac{b_{n,r}}{r+1} \omega\left(\frac{\pi}{s}\right) \right)^q \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \left[\frac{t^\gamma}{t^2 \sin^\beta \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}}$$

$$+ \frac{(n+1)^\gamma}{n+1} \left\{ \sum_{s=1}^{n} \left(\omega\left(\frac{\pi}{s}\right) \right)^q \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \left[\frac{t^\gamma}{t^2 \sin^\beta \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}}$$

$$\ll \quad (n+1)^\gamma \left\{ \sum_{s=1}^{n} \left(\sum_{r=s}^{n-1} \frac{b_{n,r}}{r+1} \omega\left(\frac{\pi}{s}\right) \right)^q s^{(2+\beta-\gamma)q-2} \right\}^{\frac{1}{q}}$$

$$+ \frac{(n+1)^\gamma}{n+1} \left\{ \sum_{s=1}^{n} \left(\omega\left(\frac{\pi}{s}\right) \right)^q s^{(2+\beta-\gamma)q-2} \right\}^{\frac{1}{q}}$$

$$\leq \quad (n+1)^\gamma \left\{ \sum_{s=0}^{n} \sum_{r=s}^{n-1} \frac{b_{n,r}}{(r+1)^q} \left(\omega\left(\frac{\pi}{s+1}\right) (s+1)^{2+\beta-\gamma-2/q} \right)^q \right\}^{\frac{1}{q}}$$

$$+ \frac{(n+1)^\gamma}{n+1} \left\{ \sum_{s=0}^{n} \left(\omega\left(\frac{\pi}{s+1}\right) (s+1)^{2+\beta-\gamma-2/q} \right)^q \right\}^{\frac{1}{q}}$$

$$\leq \quad (n+1)^\gamma \left\{ \sum_{r=0}^{n} b_{n,r} \frac{1}{(r+1)^q} \sum_{s=0}^{r} \left(\omega\left(\frac{\pi}{s+1}\right) (s+1)^{(\beta-\gamma)+2/p} \right)^q \right\}^{\frac{1}{q}}$$

$$+ (n+1)^\gamma \left\{ \frac{1}{(n+1)^q} \sum_{s=0}^{n} \left(\omega\left(\frac{\pi}{s+1}\right) (s+1)^{(\beta-\gamma)+2/p} \right)^q \right\}^{\frac{1}{q}}$$

$$\leq \quad (n+1)^\gamma \left\{ \sum_{r=0}^{n} b_{n,r} \frac{1}{r+1} \sum_{s=0}^{r} \left(\omega\left(\frac{\pi}{s+1}\right) (s+1)^{(\beta-\gamma)+1/p} \right)^q \right\}^{\frac{1}{q}}$$

$$+ (n+1)^\gamma \left\{ \frac{1}{n+1} \sum_{s=0}^{n} \left(\omega\left(\frac{\pi}{s+1}\right) (s+1)^{(\beta-\gamma)+1/p} \right)^q \right\}^{\frac{1}{q}}.$$

Further, similarly as in the estimates of I_2

$$
\begin{aligned}
I_3 &\ll \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)|}{t^2}\left[\sum_{r=\tau}^{n} \frac{b_{n,r}}{r+1}+\frac{1}{n+1}\right] dt. \\
&\leq (n+1)^{\gamma}\left\{\sum_{r=0}^{n} b_{n,r} \frac{1}{r+1} \sum_{s=0}^{r}\left(\omega\left(\frac{\pi}{s+1}\right)(s+1)^{\beta-\gamma+1/p}\right)^{q}\right\}^{\frac{1}{q}} \\
&\quad + (n+1)^{\gamma}\left\{\frac{1}{n+1} \sum_{s=0}^{n}\left(\omega\left(\frac{\pi}{s+1}\right)(s+1)^{\beta-\gamma+1/p}\right)^{q}\right\}^{\frac{1}{q}}.
\end{aligned}
$$

If $\beta-\gamma>0$, then

$$
\begin{aligned}
&|T_{n,A,B} f(x)-f(x)| \\
\ll\ & (n+1)^{\gamma+1} \omega\left(\frac{\pi}{n+1}\right)\left\{\sum_{r=0}^{n} b_{n,r} \frac{1}{r+1} \sum_{s=0}^{r}(s+1)^{(\beta-\gamma) q-1}\right\}^{1/q} \\
& + (n+1)^{\gamma+1} \omega\left(\frac{\pi}{n+1}\right)\left\{\frac{1}{n+1} \sum_{s=0}^{n}(s+1)^{(\beta-\gamma) q-1}\right\}^{\frac{1}{q}} \\
=\ & (n+1)^{\beta+1/p} \omega\left(\frac{\pi}{n+1}\right)\left\{\left[(n+1)^{1-(\beta-\gamma) q} \sum_{r=0}^{n} b_{n,r}(r+1)^{(\beta-\gamma) q-1}\right]^{1/q}+1\right\} \\
\leq\ & 2(n+1)^{\beta+1/p} \omega\left(\frac{\pi}{n+1}\right)\left[(n+1)^{1-(\beta-\gamma) q} \sum_{r=0}^{n} b_{n,r}(r+1)^{(\beta-\gamma) q-1}\right]^{1/q}
\end{aligned}
$$

Collecting these estimates we obtain the desired result. ■

3.3. Proof of Theorems 3 and 4. If we put $b_{n,r}=\frac{1}{n+1}$ in the above proofs, then the desired estimates immediately hold true. ■

3.4. Proof of Corollary 1. We have to show that the condition (2.7) and the monotonicity of (p_ν) imply (2.6). Indeed, putting

$$
a_{r,k}=\frac{p_{r-k}}{P_r}
$$

and taking $\tau=1$ in (2.7) we can see that

$$
1 \gg P_1 \sum_{\nu=1}^{r} \frac{1}{P_\nu} \geq p_0 \sum_{\nu=1}^{r} \frac{1}{P_r}=\frac{p_0}{P_r} r,
$$

whence, by the monotonicity of (p_ν) we have

$$
P_{r+1} \geq (r+1)\, p_{r+1}
$$

and therefore

$$\begin{aligned}
|a_{r,r-l}-a_{r+1,r+1-l}| &= \frac{p_l}{P_r}-\frac{p_l}{P_{r+1}}=p_l\left(\frac{1}{P_r}-\frac{1}{P_{r+1}}\right)\\
&= p_l\frac{P_{r+1}-P_r}{P_r}=\frac{p_l}{P_r}\frac{p_{r+1}}{P_{r+1}}\\
&\leq \frac{p_l}{P_r}\frac{p_{r+1}}{(r+1)\,p_{r+1}}=\frac{p_l}{(r+1)\,P_r}\\
&\leq \frac{p_0}{(r+1)\,P_r}\ll\frac{1}{(r+1)^2}.
\end{aligned}$$

Thus the desired implication follows. ■

3.5. Proofs of Theorems 5 and 6. The proofs are similar to the above. In the estimates under L^p norms with respect to x there will be the expressions like these on the left hand side of our conditions (2.1) , (2.2) and (2.5) . Since $f\in L^p(\omega)_\beta$, the such norm quantities will always have the same orders like these on the right hand side of the mentioned conditions. Therefore the proofs follow without any additionally assumptions on f and ω. ■

References

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