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Review

Dedekind cuts in $C(X)$

by
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The author considers the Dedekind cuts in the space $C(X)$ of all continuous real-valued functons on a completely regular topological space X and characterizes them as Hausdorff continuous interval-valued functions.
A *cut* in an ordered set without endpoints is a pair (A, B) of two nonempty subsets such that A coincides with the set of all lower bounds of B and, B coincides with the set of all upper bounds of A. Using A^u and A^l in order to denote the set of all upper bounds and the set of all lower bounds of A, respectively, (A, B) is a cut if $A = B^l$ and $A^u = B$ hold simultaneously. A set A is called *lower normal* if $A = A^{ul}$ and *upper normal* if $A = A^{lu}$.

An interval-function f on X assigns to each point $x \in X$ a compact interval $f(x) = [\underline{f}(x), \overline{f}(x)]$. For convenience we keep the notation $[\underline{f}, \overline{f}]$ for an interval function f. A locally bounded interval-valued function f is *Hausdorff continuous* if (i) the functions $\underline{f}$ and $\overline{f}$ are lower and upper semicontinuous, respectively, and (ii) the lower limit function[1] $I(f)$ of $\overline{f}$ is $\underline{f}$ and the upper limit function $S(f)$ of $\underline{f}$ is $\overline{f}$. The set of all Hausdorff continuous functions on X is a Dedekind complete lattice. A function $f\colon X \to \mathbb{R}$ is *C-bounded* on X if there are two continuous functions g_1, g_2 such that $g_1 \le f \le g_2$. For a C-bounded function f define for all $x \in X$ the functions

$$L(f)(x) = \sup\{g(x)\colon g \in L_f\} \quad \text{and} \quad U(f)(x) = \inf\{g(x)\colon g \in U_f\},$$

where $L_f = \{g \in C(X) : g \le f\}$ and $U_f = \{g \in C(X) : g \ge f\}$.

Let further on the topological space X be completely regular. Then the following statements are proved:
1) If f is C-bounded then $S(f) = U(f)$ and $I(f) = L(f)$ (Prop.4.1).
2) If f is C-bounded and normal upper semicontinuous[2] then U_f is an upper normal subset of $C(X)$, i.e. $\left(U_f\right)^{lu} = U_f$ (Prop.4.2).

[1]for the precise definitions see the paper

[2]i.e. $S(I(f)) = f$ for normal upper semicontinuous and, $I(S(f)) = f$ for normal lower semicontinuous f

3) If f is C-bounded and normal lower semicontinuous then L_f is an lower normal subset of $C(X)$, i.e. $\left(L_f\right)^{ul} = L_f$ (Prop.4.2).
Conversely (Prop.4.4), for a cut (A, B) in $C(X)$ with $\underline{f} = \sup A$ and $\overline{f} = \inf B$ the following statements hold:
4) $A^u = U_{\underline{f}}$, $(U_{\underline{f}})^l = L_{S(\underline{f})} = A$ and $\underline{f}$ is normal lower semicontinuous
5) $B^l = L_{\overline{f}}$, $(L_{\overline{f}})^u = U_{I(\overline{f})} = B$ and $\overline{f}$ is normal upper semicontinuous.

The main result is now
Theorem 4.6. To each cut (A, B) in $C(X)$ for a completely regular space X there corresponds a C-bounded Hausdorff continuous interval-valued function $[\underline{f}, \overline{f}]$ with $\underline{f} = \sup A$ and $\overline{f} = \inf B$ and conversely, to each C-bounded Hausdorff continuous interval-valued function $[\underline{f}, \overline{f}]$ there corresponds a the cut (A, B), where $A = L_{\overline{f}}$ and $B = U_{\underline{f}}$.

The results of the paper are very interesting and useful, so they are worth for publishing in the Proceedings of the Conference.

Some remarks:
1. The original Dedekind construction of the reals by the method of cuts in the rationals is contained in the paper „Stetigkeit und Irrationale Zahlen", whose roots go back to 1858 and which can be found in the book
R. Dedekind: Was sind und was sollen Zahlen? - Stetigkeit und Irrationale Zahlen. VEB Dt. Verlag der Wiss. 8.Auflage, Berlin. 1967.
2. After the formulas (5) an (6) it should be mentioned that $\mathcal{V}_x$ denotes the sst of all neighborhoods of the point x in the space X.
3. For completeness it would be reasonable to mention the book
B.Z. Vulikh: Introduction to the Theory of Partially Ordered Spaces. Wolters-Noordhoff, Groningen, 1967 (In Russian: Fiz.-mat. Gos. Izd. 1961, Moskva), which not only contains the McNeille's construction for the order embedding of an partially ordered set into a complete lattice which preserves all suprema and infima, but also a construction of the Dedekind completion for an Archimedian vector lattice using a similar method.
4. P.9, 2.line from below, there should be: The difference is that now it is known that ....

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