****
BANACH CENTER PUBLICATIONS, VOLUME **
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 201*

JÓZEF MARCINKIEWICZ (1910–1940) – ON THE CENTENARY OF HIS BIRTH

LECH MALIGRANDA

Department of Engineering Sciences and Mathematics
Luleå University of Technology, SE-971 87 Luleå, Sweden
E-mail: lech@sm.luth.se, lech.maligranda@ltu.se

Abstract. Józef Marcinkiewicz's (1910–1940) name is not known by many people, except maybe a small group of mathematicians, although his influence on the analysis and probability theory of the twentieth century was enormous. This survey of his life and work is in honour of the 100^{th} anniversary of his birth and 70^{th} anniversary of his death. The discussion is divided into two periods of Marcinkiewicz's life. First, 1910–1933, that is, from his birth to his graduation from the University of Stefan Batory in Vilnius, and for the period 1933–1940, when he achieved scientific titles, was working at the university, did his army services and was staying abroad. Part 3 contains a list of different activities to celebrate the memory of Marcinkiewicz. In part 4, scientific achievements in mathematics, including the results associated with his name, are discussed. Marcinkiewicz worked in functional analysis, probability, theory of real and complex functions, trigonometric series, Fourier series, orthogonal series and approximation theory. He wrote 55 scientific papers in six years (1933–1939). Marcinkiewicz name in mathematics is connected with the Marcinkiewicz interpolation theorem, Marcinkiewicz spaces, the Marcinkiewicz integral and function, Marcinkiewicz–Zygmund inequalities, the Marcinkiewicz–Zygmund strong law of large numbers, the Marcinkiewicz multiplier theorem, the Marcinkiewicz–Salem conjecture, the Marcinkiewicz theorem on the characteristic function and the Marcinkiewicz theorem on the Perron integral. Books and papers containing Marcinkiewicz's mathematical results are cited in part 4 just after the discussion of his mathematical achievements. The work ends with a full list of Marcinkiewicz scientific papers and a list of articles devoted to him.

2010 *Mathematics Subject Classification*: Primary 01A70, 46E30, 46B70, 28A15, 60E15, 46B09, 60E10, 60F15, 42B15, 26D05, 01A70; Secondary: 41A10, 26D15, 01A60.

Key words and phrases: Marcinkiewicz interpolation theorem, Marcinkiewicz spaces, Marcinkiewicz integral, independent random variables, inequalities in probability, characteristic functions, differentiation theory, maximal functions, multipliers, Marcinkiewicz sets, convergence of Riemann sums, vector-valued inequalities, Fourier and orthogonal series, Lagrange interpolation, inequalities for trigonometric functions and polynomials, rearrangements of series, universal primitive functions, Perron integral.

The paper is in final form and no version of it will be published elsewhere.

1. Life of Marcinkiewicz from the birth to university (1910–1933). Józef Marcinkiewicz was born on 12^{th} April 1910 (30^{th} March 1910 in old style=Julian calendar) in the small village Cimoszka near Białystok (Poland). His parents were Klemens Marcinkiewicz (1866–1941) and Aleksandra Marcinkiewicz née Chodakiewicz (1878–1941). Józef was the fourth of five children.

The children of Klemens and Aleksandra Marcinkiewicz were: Stanisława Marcinkiewicz–Lewicka (1903–1988), Mieczysław Marcinkiewicz (1904–1976), Edward Marcinkiewicz (1908–1985), Józef Marcinkiewicz (1910–1940), Kazimierz Marcinkiewicz (1913–1946).

Photo 1. Józef Marcinkiewicz and his signature

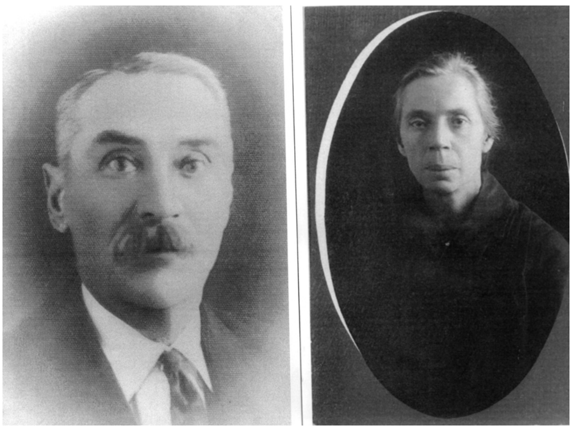


Photo 2. Parents of Józef Marcinkiewicz

Marcinkiewicz grew up with some health problems, in particular he had lung trouble, but this did not prevent him taking an active part in sports. Swimming and skiing were two sports at which he became particularly proficient. Because of his poor health, Marcinkiewicz first took private lessons at home and then he finished elementary school in Janów.

After that Marcinkiewicz went to District Gymnasium in Sokółka (after 4th class of elementary school and examination). In the period 1924–1930 he studied at the State Gymnasium of the King Zygmunt August in Białystok. He obtained his secondary-school certificate (matura certificate) on 22 June 1930 (number 220/322).

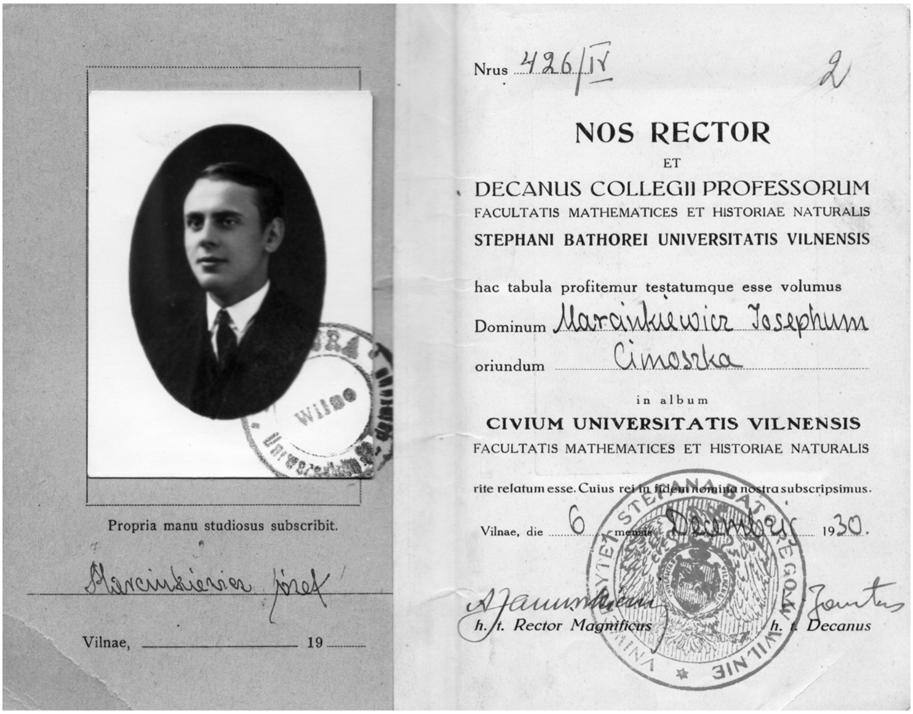


Photo 3. Photo and the first page of Marcinkiewicz's gradebook ("indeks")

In 1930 Marcinkiewicz became student at the Department of Mathematics and Natural Science, University of Stefan Batory (USB) in Wilno (then in Poland, now Vilnius in Lithuania). From the first year at the University, Marcinkiewicz demonstrated knowledge of the subject and exceptional mathematical talent. He attracted the attention of the following three professors from the Department: Stefan Kempisty, Juliusz Rudnicki and Antoni Zygmund.[1]

In his second year of studies he participated, in the academic year 1931/32, at Zyg-

[1]Stefan Jan Kempisty (born 23 July 1892 in Zamość – died 5 August 1940 in prison in Wilno), Juliusz Rudnicki (born 30 March 1881 in village Siekierzyńce near Kamieńec Podolski – died 26 February 1948 in Toruń), Antoni Zygmund (born 25 December 1900 in Warsaw – died 30 May 1992 in Chicago).

mund's course on *orthogonal series* proceded by an *introduction to the theory of Lebesgue integration*. This course was too difficult for the average second year student and Marcinkiewicz asked Zygmund for permission to take this course. That was the beginning of their fruitful mathematical collaboration.

As well as Marcinkiewicz, there were a few other young mathematicians, for example, Konstanty Sokół-Sokołowski, later he obtained PhD in mathematics and senior assistant at the Department of Mathematics, who became like Marcinkiewicz one of the victims of the war (and like Marcinkiewicz he was killed in Kharkov in 1940).[2]

Zygmund wrote ([JMCP], pp. 2–3):

> *When I think of Marcinkiewicz I see in my imagination a tall and handsome boy, lively, sensitive, warm and ambitious, with a great sense of duty and honor. He did not shun amusement, and in particular was quite fond of dancing and the game of bridge. His health was not particularly good; he had weak lungs and had to be careful of himself. He was interested in sports (possibly because of his health) and was a good swimmer and skier. He also had intellectual interests outside Mathematics, knew a lot of modern Physics and certain branches of Celestial Mechanics. He said to me once that before entering the university he had hesitated about whether to choose mathematics or Polish literature.*

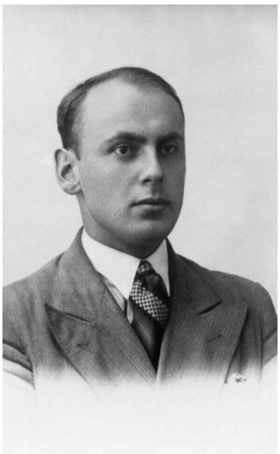


Photo 4. Józef Marcinkiewicz

We must remember that Antoni Zygmund was not only a master for Marcinkiewicz, but also an advocate for his achievements. In 1940 Zygmund emigrated to the United States, and from 1947 he worked in Chicago, where he created the famous school of mathematics. His students were among others Alberto Calderón (1920–1998), Leonard

[2]Konstanty Sokół-Sokołowski (born 9 September 1906 in Tarnobrzeg – killed in April or May 1940 in Kharkov). He has written his PhD thesis under supervision of Zygmund in 1939 *On trigonometric series conjugate to Fourier series of two variables.*

D. Berkovitz (1924–2009), Paul J. Cohen (1934–2007) – awarded the Fields Medal in 1966, Misha Cotlar (1912–2007), Eugene Fabes (1937–1997), Nathan Fine (1916–1994), Benjamin Muckenhoupt (1933), Victor L. Shapiro (1924), Elias Stein (1931), Daniel Waterman (1927), Guido Weiss (1928), Mary Weiss (1930–1966), Richard Wheeden (1940) and Izaak Wirszup (1915–2008). It is thanks to Zygmund, who survived the war and became an important mathematician in the world, the name of Marcinkiewicz also became widely known among mathematicians.

Marcinkiewicz was interested in literature, music, painting, poetry, and he also wrote poetry himself. He liked almost all areas of life and also to talk about different topics. While studying he learnt English, French and Italian. Mathematics, however, he always put first. He took an active part in the student life participating in various events organized by the Mathematical-Physical Circle; in the academic year 1932/33 he was president of the Board. His closer friends and colleagues were: Stanisław Kolankowski, Wanda Onoszko, Danuta Grzesikowska(-Sadowska) and Leon Jeśmanowicz.

Marcinkiewicz graduated in 1933, after only three years of study and on 20 June 1933 he obtained a Master of Science degree (in mathematics) at USB. The title of his master thesis was *Convergence of the Fourier–Lebesgue series* and the supervisor was Professor Antoni Zygmund.

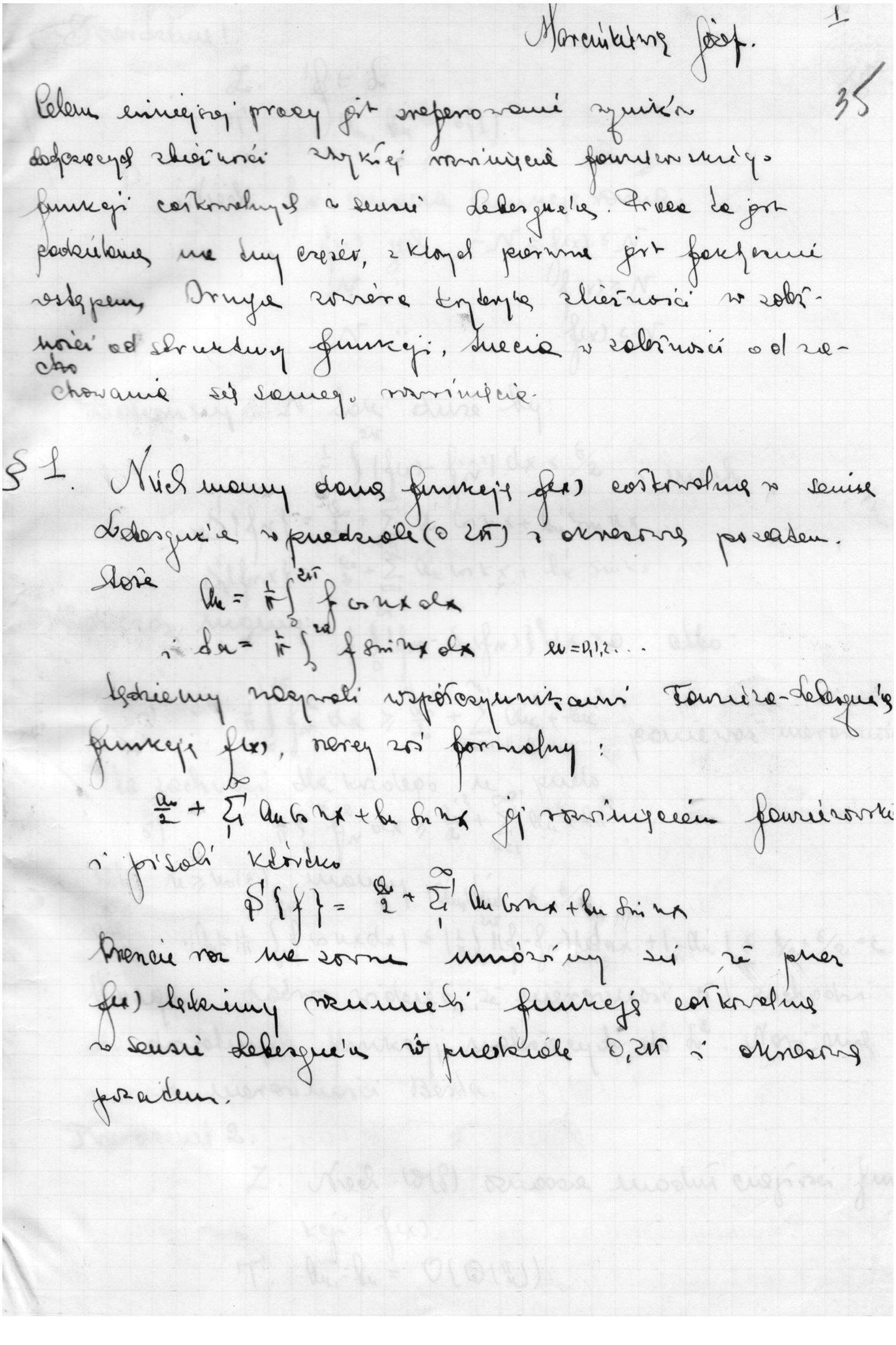
His M.Sc. thesis consisted of his first original results in mathematics and contained, among other things, the proof of the new and interesting theorem that there exists a continuous periodic function whose trigonometric interpolating polynomials, corresponding to equidistant nodal points, diverge almost everywhere. These results, in a somewhat extended form, were presented two years later as his PhD thesis.

2. Scientific career, work, military service and tragic end. In the periods September 1933 – August 1934 and September 1934 – August 1935 he was assistant to the Zygmund chair of mathematics at USB.

In the interim he did one year military service in 5$^{\text{th}}$ Infantry Regiment of Legions in Wilno. He finished the military course with an excellent score. On 17$^{\text{th}}$ September 1934 he was transferred to the reserves. Marcinkiewicz took his soldiering duties seriously, but not without a sense of humour as far as the disadvantages of military service were concerned. He received the following evaluation:

> *Outstanding individuality. Very energetic and full of initiative.* (...) *Deep and bright mind.* (...) *Memory and logical thinking very good* (...) *Characterized by a good planning and persistence in work. Overall evaluation: outstanding.*

In September 1934 he returned to USB and on 25$^{\text{th}}$ June 1935 he defended his PhD thesis entitled *Interpolation polynomials of absolutely continuous functions* at the University of Stefan Batory in Wilno, under supervision of Antoni Zygmund. His PhD thesis was a booklet of 41 pages published in Polish in Dissertationes Inaugurales No. 10, USB. It was also published as the paper in [M35c] and its English translation appeared in [JMCP], pp. 45–70.


Marcinkiewicz Józef. 1

35

Celem niniejszej pracy jest zreferowanie wyników dotyczących zbieżności zwykłej rozwinięcia fourierowskiego funkcji całkowalnych w sensie Lebesgue'a. Praca ta jest podzielona na trzy części, z których pierwsza jest jedynie wstępem. Druga zawiera kryteria zbieżności w zależności od struktury funkcji, trzecia w zależności od zachowania się samego rozwinięcia.

§ 1. Niech mamy daną funkcję f(x) całkowalną w sensie Lebesgue'a w przedziale (0 2π) i okresową poza tem. Stałe

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f\cos nx\,dx$$

$$\text{i } b_n = \frac{1}{\pi}\int_0^{2\pi} f\sin nx\,dx \qquad n = 0, 1, 2 \ldots$$

będziemy nazywali współczynnikami Fourier-Lebesgue'a funkcji f(x), szereg zaś formalny:

$$\frac{a_0}{2} + \sum_1^\infty a_n\cos nx + b_n\sin nx$$ jej rozwinięciem fourierowskiem i pisali krótko

$$S\{f\} = \frac{a_0}{2} + \sum_1^\infty a_n\cos nx + b_n\sin nx$$

Wreszcie raz na zawsze umówimy się, że przez f(x) będziemy rozumieli funkcję całkowalną w sensie Lebesgue'a w przedziale 0,2π i okresową poza tem.

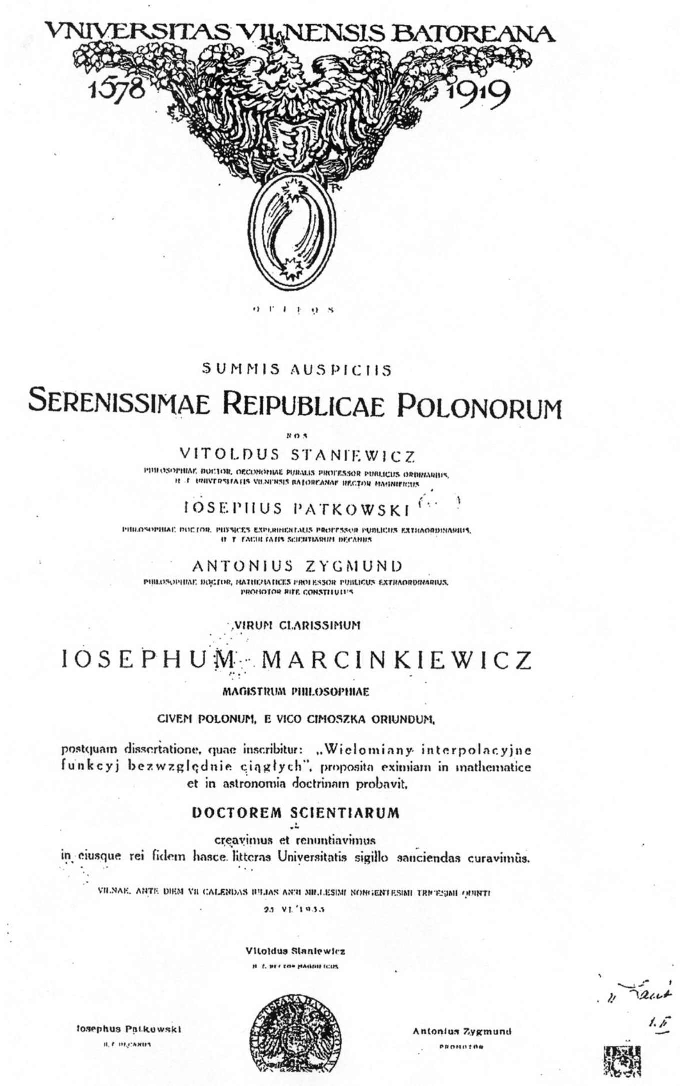
Photo 5. 1933. The first page of hand-written master thesis of Józef Marcinkiewicz

Photo 6. XI Congress of Scientific Mathematical-Physical and Astronomical Circles (25–28 May 1933) in Wilno. Congress Hall USB. Siting in the first row from the left: Kazimierz Jantzen (1885–1940), Józef Stanisław Patkowski (1887–1942) – physicist, Kazimierz Opoczyński (1877–1963) – rector of USB, Władysław Dziewulski (1878–1962) – astronomer, Józef Marcinkiewicz (1910–1940), Ira Anna Koźniewska (1911–1989) – statistician, Wacław Michał Dziewulski (1882–1938) – physicist, Stefan Jan Kempisty (1892–1940), Bogumił Jasinowski (1883–1969) – philosopher, Aleksander Januszkiewicz (1872–1955), Antoni Zygmund (1900–1992), Edward Szpilrajn-Marczewski (1907–1976)

The evaluation of Marcinkiewicz's PhD dissertation made by Zygmund (28 May 1935) contains the following opinion:

> *I think Marcinkiewicz's work is very valuable, showing big mathematical talent and originality of the author. I accept it as a doctoral dissertation.*

At the PhD examination, taken on 7$^{\text{th}}$ June 1935, the questions to Marcinkiewicz were the following: Zygmund: *problem of approximation of functions, interpolation theory (Legendre, Hermite), quadratic approximation, Chebyshev polynomials, Fejér results, convergence criterion in the case when nodes are zeros of Jacobi polynomials, means approximation of* $p \neq 2$; Rudnicki: *entire functions, results of Weierstrass, Poincaré, Borel, Picard, Hadamard, theory of Nevanlinna and Julia*; Kempisty: *Perron integral and its different definitions, similar question for functions of two-variables, surface and its measuring, results of Rado and Tonelli*; Dziewulski: *equations of motion in mechanics of celestial bodies, integrals of these equations, the perturbation function, Lagrangian points,* *motion of the stars and the currents of stars.*


VNIVERSITAS VILNENSIS BATOREANA

1578 1919

SUMMIS AUSPICIIS

SERENISSIMAE REIPUBLICAE POLONORUM

NOS

VITOLDUS STANIEWICZ

PHILOSOPHIAE DOCTOR, OECONOMIAE RURALIS PROFESSOR PUBLICUS ORDINARIUS, H. T. UNIVERSITATIS VILNENSIS BATOREANAE RECTOR MAGNIFICUS

IOSEPHUS PATKOWSKI

PHILOSOPHIAE DOCTOR, PHYSICES EXPERIMENTALIS PROFESSOR PUBLICUS EXTRAORDINARIUS, H. T. FACULTATIS SCIENTIARUM DECANUS

ANTONIUS ZYGMUND

PHILOSOPHIAE DOCTOR, MATHEMATICES PROFESSOR PUBLICUS EXTRAORDINARIUS, PROMOTOR RITE CONSTITUTUS

VIRUM CLARISSIMUM

IOSEPHUM MARCINKIEWICZ

MAGISTRUM PHILOSOPHIAE

CIVEM POLONUM, E VICO CIMOSZKA ORIUNDUM,

postquam dissertatione, quae inscribitur: „Wielomiany interpolacyjne funkcyj bezwzględnie ciągłych", proposita eximiam in mathematice et in astronomia doctrinam probavit,

DOCTOREM SCIENTIARUM

creavimus et renuntiavimus
in eiusque rei fidem hasce litteras Universitatis sigillo sanciendas curavimus.

VILNAE, ANTE DIEM VII CALENDAS IULIAS ANNI MILLESIMI NONGENTESIMI TRICESIMI QUINTI

25 VI 1935

Vitoldus Staniewicz

H. T. RECTOR MAGNIFICUS

Iosephus Patkowski

H. T. DECANUS

Antonius Zygmund

PROMOTOR

Photo 7. Diploma of doctor of philosophy

During the years 1934–1938 Marcinkiewicz was taken on six-weeks military exercises (25$^{\text{th}}$ June–16$^{\text{th}}$ September 1934, 12$^{\text{th}}$ August–21st September 1935, 1st July–10$^{\text{th}}$ August 1936, 1938 – before travel to France). The academic year 1935/1936 he spent at the University of Jan Kazimierz in Lwów. This was a one year Fellowship from the Fund for National Culture and the assistant position at the chair of Stefan Banach in the period 1 December 1935–31 August 1936 with 12th hours of teaching weekly (cf. [DP10], pp.

59–61). Marcinkiewicz visited the *Scottish Café*. He solved problems 83 of Auerbach, 106 of Banach and 131 of Zygmund from the *Scottish Book*. Moreover, he posed his own problem number 124 (cf. [Ma81], pp. 211–212). In Lwów Marcinkiewicz cooperated with Juliusz Paweł Schauder (1899–1943), who had returned to Lwów a year earlier having spent time in Paris working with Hadamard and Leray.[3]

Photo 8. Józef Marcinkiewicz

Zygmund writes ([JMCP], p.3):

> *The influence of Schauder was particularly beneficial and would probably have led to important developments had time permitted. For in the field of real variable Marcinkiewicz had exceptionally strong intuition and technique, and the results he obtained in the theory of conjugate functions, had they been extended to functions of several variables might have given (as we see clearly now) a strong push to the theory of partial differential equations. The only visible trace of Schauder's influence is a very interesting paper of Marcinkiewicz on the multipliers of Fourier series, a paper which originated in connection with a problem proposed by Schauder* (...)

While in Lwów Marcinkiewicz also collaborated with Stefan Kaczmarz (1895–1939) and Władysław Orlicz (1903–1990)[4] Marcinkiewicz became interested in problems of general orthogonal systems and wrote a series of papers on this subject. He published joint paper with Kaczmarz on multipliers of Fourier series and was working in Lwów on general orthogonal series.

Marcinkiewicz was nominated senior assistant to the chair of mathematics at USB for the period 1^{st} September 1936 – 31^{st} August 1937 and on 16^{th} April 1937 Marcinkiewicz filled in an application to commence his habilitation. After one month, on 25^{th} May 1937

[3]Juliusz Paweł Schauder (born 21 September 1899 in Lwów – killed in September 1943).

[4]Stefan Kaczmarz (born 20 March 1895 in Sambor – killed in September 1939), Władysław Orlicz (born 24 May 1903 in Okocim – died 9 Aug. 1990 in Poznań).

Zygmund wrote the following opinion about the papers of Marcinkiewicz:

> *From the above discussion the work of Dr. Marcinkiewicz shows that it contains a number of interesting and important results. Some of them, due to their final form, will certainly appear in textbooks in mathematics. It should be mentioned that in some of the early papers we can already see strong and subtle arithmetic techniques; things of rare quality. The entire collection is extremely favorable and testifies to the multilateral and original mathematical talent of the author.*

Photo 9. Wilno, 4 March 1936. Doctor *honoris causa* for Professor Kazimierz Sławiński[5]. In the foreground (from the left): Juliusz Rudnicki, N.N., Kornel Michejda, Stefan Kempisty, Edward Bekier. At the wall (fourth and fifth from the left): Józef Marcinkiewicz and Antoni Zygmund

Marcinkiewicz's habilition discussion (exam) was taken on 11^{th} June 1937 and the questions raised included the following: Zygmund: *1. Unsolved questions in the theory of trigonometric series, orthogonal series and interpolational polynomials, 2. Questions connected with the Laplace–Lyapunov theorem*; Rudnicki: *Integral equations*; Kempisty: *Generalizations of the integral concept.*

After the procedure, on 12^{th} June 1937 his habilitation *On summability of orthogonal series* was approved and the nomination to docent by USB was given. His habilitation lecture had the title *Arithmetization of notion of eventual variable.* The second proposed topic was *Convergence of interpolational polynomials.*

[5]Kazimierz Sławiński (1871–1941), chemist, professor of the University of Stefan Batory in Wilno.

At the age 27 Marcinkiewicz was the youngest doctor with habilitation at the University of Stefan Batory. The same year Marcinkiewicz was awarded the Scientific Prize of J. Piłsudski.

His senior assistant position at USB was prolonged on the periods 1^{st} September 1937–31^{st} August 1938 and 1^{st} September 1938–31^{st} August 1939. In the meantime, he participated in the 3^{rd} Congress of Polish Mathematicians (29 September–2 October 1937) in Warsaw with the lecture *On one-sided convergence of orthogonal series*. In January 1938 he was nominated for lieutenant reserve, where we can find information about him: height – 180 centimeters, hair – dark blond, eye colour – bright hazel. The same year he received one year scholarship from the Fund for National Culture for the trips to Paris, London and Stockholm to complement his knowledge in probability theory and mathematical statistics. On 9^{th} July 1938 the Ministry of Religious Creeds and Public Education granted Marcinkiewicz paid leave for the academic year 1938/1939.

Photo 10. Józef Marcinkiewicz

On 11^{th} October 1938 Marcinkiewicz presented a talk in Poznań *The development of the probability theory for the last 25 years*. This lecture was probably connected with his application for a professor position in Poznań.

After this visit he went to Paris, where he stayed six months (October 1938–March 1939). In this period Marcinkiewicz collaborated with Stefan Bergman and Raphaël Salem.[6] With Bergman he has written two joint papers in the theory of complex functions of two-variables and with Salem one paper on Riemann sums.

Marcinkiewicz also had contact with the famous mathematician Paul Lévy.[7] Bernard

[6]Stefan Bergman (born 5 May 1895 in Częstochowa – died 6 June 1977 in Palo Alto, California) whose name, in two joint papers with Marcinkiewicz, is written with two “n” at the end, i.e., as Bergmann; Stefan Raphaël Salem (born 7 November 1898 in Saloniki – died 20 June 1963 in Paris).

[7]Paul Pierre Lévy (born 15 September 1886 in Paris – died 15 December 1971 in Paris), French mathematician, professor of analysis at École Politechnique in Paris from 1920 to 1959,

Bru in a conversation with Murad Taqqu discussed about contribution of Loius Bachelier (1870–1946) to Brownian motion and informed how Paul Lévy get interested in Brownian motion (see [Ta01]):

> *Lévy began to take an interest in Brownian motion toward the end of the 1930s by way of the Polish school, in particular Marcinkiewicz who was in Paris in 1938. He rediscovered all of Bachelier's results which he had never really seen earlier. Lévy had become enthralled with Brownian motion.*

We should mention here that Marcinkiewicz have sent his paper on Brownian motion [M38–40] to the journal already in January 1938 and Lévy published in 1939 paper [Le39] on a problem of Marcinkiewicz. There are also theorems with the Lévy and Marcinkiewicz names in it (cf. our sections 4.2.3 and 4.6.4).

At the end of 1938 Irena Sławińska arrived to Paris, on 9 months' scientific stay, alumnus of the Polish and Roman literature of USB in Wilno, regarded as his fiancée[8] She returned to Wilno in August 1939. After the War she was working in Toruń and from 1949 in Lublin (Catholic University of Lublin). I met her in Warsaw on 6 March 2002. She mentioned to me that they were planning to get married and also that:

> *Marcinkiewicz was going out in the middle of the film and was saying that he had no time for entertainment or this is a waste of his time on such a bad movie.*

According to K. Dąbrowski and E. Hensz-Chądzyńska ([DH02], p. 3): *During his stay in Paris he was offered a professorship in one of the American universities. He declined as he had already accepted another offer from the University of Poznań in Poland.* However, I am not aware of any documentary evidence of this American offer.

After Paris Marcinkiewicz arrived to London on his scientific stay. He was staying five months (April–August 1939) at the University College London (UCL). He managed to present his own work in Cambridge, presumably visiting J. E. Littlewood (G. H. Hardy was also there), and in Oxford (cf. [DP10], p. 27). A planned trip to Stockholm has not been realized, since Marcinkiewicz returned from London to Wilno in August 1939.

In June 1939 he became an extraordinary professor at the University of Poznań. From the new academic year 1939/40 Marcinkiewicz should take up the chair of mathematics at the University of Poznań, after Zdzisław Krygowski (1872–1955), who retired in 1938. There is even a protocol of the 8^{th} meeting of the Senate of the University of Poznań from 23 June 1939, where the point 11 is about appointment of docent Marcinkiewicz on associate professor of mathematics and this was presented by the Dean Suszko.[9] Unfortunately, the outbreak of war, disrupted his plans. By the way, Marcinkiewicz on 27^{th} June 1939 was also nominated senior assistant to the mathematics chair at USB for the period 1^{st} September 1939–31^{st} August 1940.

---

who introduced in 1922 the term *functional analysis*. The author of 10 books and over 250 papers in probability, functional analysis and partial differential equations.

[8] Irena Zofia Sławińska (born 30 August 1913 in Wilno – died 18 January 2004 in Warsaw).

[9] Jerzy Suszko (1889–1972), chemist, from 1937 Professor of the University of Poznań.

At the end of August 1939 Marcinkiewicz returned to Wilno from London. In the second half of August 1939, Marcinkiewicz stayed in England. The outbreak of war was imminent. In Poland, general mobilisation was announced. Despite his colleagues advice to stay in England, he decided to go back to Poland. Marcinkiewicz was answering to them that (cf. [MM76], p. 16):

> *as a patriot and son of my homeland would never attempt to refuse the service to the country in such difficult time as war.*

Norman L. Johnson[10] in a conversation with C. B. Read in May 2002 was saying (cf. [Re04], p. 557):

> *I would like to mention the influence that someone had on my life to some extent. In my first, prewar, year on staff at University College, we had a visitor from Poland, Jozef Marcinkiewicz. He was only over for a month or so because he was also visiting Paris. He was a very good theoretical probabilist, he was interested in statistics and he was very remarkable. He was only 28 years old and already a Professor in Poland. We had a lot of talks. I was flattered that he took notice of such a junior member of the staff as I was, in my corner of the laboratory where I felt protected against Fisher. We talked a lot about that, and he came to me for what he called a good practical outlook, thinking that his mathematical statistics ought to be more applicable than it was in the way he had learned it. I also thought I was learning a lot more about mathematics than I had ever learned as an undergraduate in University College. We had a lot of conversations then. When he left in the spring of 1939, it was pretty clear there was going to be a war. He already had been offered a post in the United States, and I said (which almost destroyed our friendship), "Won't you perhaps accept this and be out of the way if the Germans invade Poland?" He was extremely indignant. He said, "My duty is to go back and defend my country, I am a reserve officer and I am surprised you would think of something as bad as that. Why don't you go off to the United States?" I was able to calm him down and sort that out. He did go, and he was taken prisoner by the Russians and ultimately murdered in Katyn Forest near Smolensk. I always felt that I would like to take an opportunity of saying how highly I thought of him as a person and as a probabilist who was appreciative of statistics and that somehow or other thought I could do something useful. At that time when I was just starting, as you know, you are not very sure of anything. I would like to take the opportunity of mentioning that. In fact, in the Encyclopedia, there is an entry "Marcinkiewicz's Theorem." He wrote a book, but I don't know the title.*

This shows that Marcinkiewicz was not only scientist, but he was also a great Polish

[10]Norman Lloyd Johnson (1917–2004) completed his M.Sc. in statistics in 1938 under supervision of Jerzy Neyman at the Department of Applied Statistics at University College London. The same year he joined the same Department as an Assistant Lecturer on an invitation by Egon Pearson. From 1962 he became Professor at the University of North Carolina at Chapel Hill (USA). He was the author and co-author of 17 books and more than 180 papers in statistics.

patriot returning to Poland. He could have stayed in England or go to United States, but in his opinion, this meant desertion. Instead, he chose to fight and thus became a martyr.

Zygmund writes ([Zy64], p. 4):

> *On September 2nd, the second day of the war, I came across him accidently in the street in Wilno, already in military uniform (he was an officer of the reserve). We agreed to meet the same day in the evening but apparently circumstances prevented him from coming since he did not show up at the appointed place. A few months later came the news that he was a prisoner of the war and was asking for mathematical books. It seems that this was the last news about Marcinkiewicz.*

and in [Z751, p. 8]:

> *Marcinkiewicz, mobilized, was taken prisoner and disappeared without trace.*

Marcinkiewicz was a reserve officer assigned to the 2nd Battalion, 205th Infantry Regiment, and took part in the defence of Lwów (12–21 September 1939). After capitulation of Lwów (22 September 1939) he, together with other officers, was taken prisoner by the Red Army.

Photo 11. Józef Marcinkiewicz in uniform (second from the left).
First from the right – his brother Edward

Stanisława Lewicka (sister of Józef) wrote on 12 October 1959 in the letter to Wiadomości Matematyczne [Mathematical News] ([Le1959], pp. 1–2):

> *During his time in Paris and England, Marcinkiewicz had produced some mathematical work, which he had written down in manuscript form. After returning to Poland he gave these manuscripts to his parents for safe keeping. Sadly Marcinkiewicz's parents suffered the same fate as he did and died*

> *in June 1941 in Buchara (Uzbekistan). After the war his brother Kazimierz accidentally dug this work from the ground, but they were unfortunately in a state of decomposition.*

Stanisław Kolankowski wrote on Józef Marcinkiewicz:

> *We met for the first time on* 20^{th} *September. (...) At night the German army started to leave their positions, and then the Soviet Army came. The Lwów defence committee decided to give the city up to the Soviet Army. The Soviets "temporarily interned" the officers commanding the defence of Lwów. (...). It was the* 25^{th} *of September. I found out that Józef Marcinkiewicz was in the same car with me. (...). The railway workers told us we would be located in camps throughout the Soviet Union. Then I decided to flee from the transport along with two other officers from Lwów. I insisted that Marcinkiewicz go with us. He decided not to go. (...) The railway workers told us we were going to Starobielsk (a small town near Kharkov in Ukraine). Just before the Polish-Soviet border at the Podwysokie station all three of us jumped off the train. I saw Józef Marcinkiewicz at ten p.m. for the last time.*

Marcinkiewicz was kept in the Starobielsk camp from September 1939 until April or May 1940 (registered under the id number 2160; victim index number 6444). The family had received two postcards from the Starobielsk camp. The last one was dated March 1940. Marcinkiewicz also sent some postcards and letters from Starobielsk to his close friends, including Zygmund and Jeśmanowicz.

It is known description of stay of Marcinkiewicz in the Starobielsk camp, written by Zbigniew Godlewski[11] school colleague, and published in the Review of History (Przegląd Historyczny 38(1993), z. 2, pp. 323–324) in 1992 *Lived through Starobielsk*.

Probably, the Soviets soon discovered how brilliant their captive was. They offered him some form of collaboration. Marcinkiewicz allegedly asked in a letter for his mathematical books and a copy of his PhD certificate to be sent to him at the camp. It is supposed that, in the end, Marcinkiewicz declined the Soviet offer.

Marcinkiewicz was then murdered in Kharkov, where thousands of Polish officers were executed. He is probably buried in the village Piatykhatky (Piatichatki).

The exact date of Józef Marcinkiewicz's death remains unknown because some official Soviet documents are inaccessible or have been destroyed. The only known information is that this was between 5 April and 12 May 1940.

At the cemetery situated in Janów there is a grave containing the ashes of Kazimierz Marcinkiewicz. A plaque commemorates also Józef Marcinkiewicz and his parents Aleksandra and Klemens Marcinkiewicz. This plaque was founded by the priest Józef Marcinkiewicz in 1956. The plaque bears the names of the parents of Józef, the name of Józef and his brother Kazimierz; above that there are words:

In honour of the martyrdom of the Marcinkiewicz family.

---

[11] Zbigniew Godlewski (1909–1993), from 3 October 1939 prisoner of Starobielsk camp. Memories were written in 1980.

This inscription gives the most tragic fate of the Polish family during the war. Marcinkiewicz's parents Klemens and Aleksandra were transported in June 1941 to Uzbekistan by the NKVD and six months later they died of hunger in Buchara on 24 December 1941. Józef was exectuted in Kharkov in Spring 1940. Edward, who was later transported to Siberia, joined the Polish army of General Anders and took part in the battle of Monte Cassino (Italy). He then lived in Argentina, Italy and Switzerland. The youngest brother, Kazimierz, one of the defenders of Lwów, returned to his family's house. Like Mieczysław and Stanisława, he was a member of the Polish underground during the Soviet and Nazi occupation and at the beginning of the Communist regime. In 1946 he was killed by security officers in Janów. Mieczysław was forced by communist authorities to sell the farm and move to a different place (Krapkowice).

Photos 12–13. Symbolic grave of Marcinkiewicz family in Janów

Józef Marcinkiewicz should be remembered as a true Polish patriot, and especially as an outstanding mathematician. Mathematics was his passion. He possessed an outstanding ability to focus on problems in mathematical thinking and had extraordinary insights in mathematics. Marcinkiewicz's premature death was a huge blow to Polish and world mathematics.

Zygmund, in his article about Marcinkiewicz, wrote ([Zy64], p. 1):

> *His first mathematical paper appeared in 1933; the last one he sent for publication in the Summer of 1939. This short period of mathematical activity left, however, a definite imprint on Mathematics, and but for his premature*

death he would probably have been one of the most outstanding contemporary mathematicians. Considering what he did during his short life and what he might have done in normal circumstances one may view his early death as a great blow to Polish Mathematics, and probably its heaviest individual loss during the second world war.

3. Contests, books, conferences, lectures, exhibitions, awards and special lectures dedicated to Marcinkiewicz. Books, competitions, conferences, exhibitions and awards in his name have since celebrated the memory of Marcinkiewicz. They are presented below in chronological order with a short description.

1947. Poznań. In October 1947 Polish Mathematical Society in Poznań intended to publish *Commemorative Book*, and Dr. Andrzej Alexiewicz (later professor of mathematics) has asked the Stanisława Lewicka – sister of Józef Marcinkiewicz for biographical material. Unfortunately, the *Book* did not appear in print.

1957. Toruń. In memory of Marcinkiewicz Toruń Branch of the Polish Mathematical Society initiated in 1957 an annual Marcinkiewicz's competition for the best student's mathematical paper. The winners of the first competition in 1957 were: Z. Ciesielski, K. Sieklucki, A. Schinzel and A. Jankowski.

1959. New York. Antoni Zygmund dedicated his famous monograph *Trigonometric Series*, 2nd ed., Cambridge University Press, New York 1959 in the following way: *Dedicated to the memories of A. Rajchman and J. Marcinkiewicz. My teacher and my pupil.*

1960. Warsaw. On the twentieth anniversary of the death of Józef Marcinkiewicz the Mathematical News (Wiadomości Matermatyczne) published in Polish an article of Zygmund on Marcinkiewicz: A. Zygmund, *Józef Marcinkiewicz*, Wiadom. Mat. (2) 4(1960), 11–41.

1964. Warsaw. In recognition of the great mathematical achievements of Marcinkiewicz, Polish Academy of Sciences edited in 1964, on 681 pages, his collected papers: *Józef Marcinkiewicz, Collected Papers*, PWN, Warsaw 1964. Only a few Polish mathematicians have been honored in this way.

1980. Warsaw. Comission of the History of Mathematics of the Polish Mathematical Society and Institute of History of Science of the Polish Academy of Sciences organized on 11 December 1980 a scientific session dedicated to Józef Marcinkiewicz on the occasion of his 70th anniversary of birth. Two lectures were given: Z. Ciesielski, *The scientific output of Józef Marcinkiewicz*, L. Jeśmanowicz, *Previous history of the Marcinkiewicz's competition*. Moreover, L. Jeśmanowicz was talking about J. Marcinkiewicz.

1981. Chicago. On the occasion of Zygmund's 80$^{\text{th}}$ birthday the conference *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, University of Chicago, Chicago, Ill., March 23–28, 1981 was organized. The wish of Zygmund was that *each speaker should quote or rely on some statement of Marcinkiewicz.* Proceedings of the conference were published in two volumes and edited by William Beckner, Alberto P. Calderón, Robert Fefferman and Peter W. Jones in 1983 on 852 pages.

1981. Olsztyn. The 16$^{\text{th}}$ Scientific Session of the Polish Mathematical Society (15–17 September 1981). Z. Ciesielski presented a talk *Ideas of Józef Marcinkiewicz in mathe-*

matical analysis.

1988. Katowice. Third All Polish School on History of Mathematics *Mathematics at the turn of the twentieth century*, May 1988. Lecture: B. Koszela, *The contribution of Józef Marcinkiewicz, Stefan Mazurkiewicz and Hugo Steinhaus in developing Polish mathematics. A biographical sketch* (in Polish).

1991. Dziwnów. Fifth All Polish School on History of Mathematics *Probability and mechanics in historical sketches*, 9–13 May 1991. Lectures on Marcinkiewicz: E. Hensz and A. Łuczak, *Strong law of large numbers of Marcinkiewicz, classical and non-commutative version*, E. Hensz, *Józef Marcinkiewicz.*

1995. Toruń. Scientific Session of the Polish Mathematical Society (13–15 September 1995). Lecture: K. Dąbrowski i E. Hensz, *Józef Marcinkiewicz* (*1910–1940*).

2000. Będlewo. 16–20 October 2000. Stefan Banach International Mathematical Center organized *Rajchman–Zygmund–Marcinkiewicz Symposium* dedicated to the memory of Aleksander Rajchman (1891–1940), Antoni Zygmund (1900–1992) and Józef Marcinkiewicz (1910–1940). Talk about Marcinkiewicz: K. Dąbrowski and E. Hensz-Chądzyńska, *Józef Marcinkiewicz (1910–1940). In commemoration of the* 60^{th} *anniversary of his death.*

2007. Gdańsk. Gdańsk Branch of the Polish Mathematical Society and Institute of Mathematics of the Gdańsk University organized on 29 October 2007 an exposition and scientific session to commemorate Józef Marcinkiewicz. Lectures were given by: S. Kwapień, *Józef Marcinkiewicz, Wolfgang Doeblin, two lots, similarities and differences*, Z. Ciesielski, *Some reflections on Józef Marcinkiewicz* and E. Jakimowicz, *How was exhibition dedicated to Józef Marcinkiewicz organized.*

2010. Toruń. Scientific session on the hundredth anniversary of birth of Józef Marcinkiewicz (10 March 2010). Lectures: A. Jakubowski, *J. Marcinkiewicz and his achievements in the theory of probability*, Y. Tomilov, *Selected results of J. Marcinkiewicz in the theory of functions and functional analysis.*

2010. Janów. 23 March 2010. *Hundredth anniversary of the birth of Józef Marcinkiewicz (1910–1940).* The Agricultural Education Center Complex Schools in Janów. Lectures: R. Brazis (Polish University, Wilno), *Wilno – a city enlightened in legend of Józef Marcinkiewicz*, L. Maligranda (Luleå University, Sweden), *Józef Marcinkiewicz as pupil, man and mathematician* and exposition of E. Jakimowicz, *To Józef Marcinkiewicz on the occasion of the* 100^{th} *anniversary of his birth and* 70^{th} *anniversary of his death* (continuation of the exposition from 2007).

2010. Iwonicz Zdrój. 25 May 2010. The 24^{th} School of History of Mathematics (24–28 May 2010). Lectures: S. Domoradzki and Z. Pawlikowska-Brożek, *Józef Marcinkiewicz in the light of memories*, L. Maligranda (LTU), *Józef Marcinkiewicz and his mathematical achievements.*

2010. Poznań. 28 June–2 July 2010. On the occasion of the centenary of the birth of Józef Marcinkiewicz (1910–1940) University of Adam Mickiewicz (Poznań), Institute of Mathematics of the Polish Academy of Sciences, Warsaw University, Nicolaus Copernicus University (Toruń) organized a scientific conference to commemorate one of the most eminent Polish mathematicians *The Józef Marcinkiewicz Centenary Conference (JM100).* The first plenary lecture was given by L. Maligranda, *Józef Marcinkiewicz (1910–1940)*

– on the centenary of his birth.

2010. Janów. 14 Oct. 2010. Celebration on which the name of Professor Józef Marcinkiewicz was given for the Agricultural Education Center Complex Schools in Janów [Zespół Szkół Centrum Kształcenia Rolniczego w Janowie, pow. Sokółka].

4. Mathematics of Józef Marcinkiewicz. Results proved by Marcinkiewicz are in the following areas of mathematics:

- Functional Analysis (interpolation of operators, Marcinkiewicz spaces and vector-valued inequalities)
- Probability Theory (independent random variables, Khintchine type inequalities, characteristic functions, Brownian motion)
- Theory of Real Functions
- Trigonometric Series, Power Series, Orthogonal and Fourier Series
- Approximation Theory
- Theory of Functions of Complex Variables

In a period of six years (1933–1939) Józef Marcinkiewicz wrote 55 papers (while spending one year in the army). 19 were published with co-authors (14 with A. Zygmund, 2 with S. Bergman and one with S. Kaczmarz, R. Salem, B. Jessen[12] and A. Zygmund). Despite the brevity of his period of mathematical activity, it has nonetheles left a define mark on mathematics.

A list of his published papers can be found in *Józef Marcinkiewicz, Collected Papers*, PWN, Warsaw 1964, pages 31–33 and also here in part 5, which is supplemented by his printed PhD thesis [M35b] and unknown paper [M37c] and is in chronological order.

Marcinkiewicz was also reviewer for *Zentralblatt für Mathematik und ihre Grenzgebiete (Zbl)* and *Jahrbuch über die Fortschritte der Mathematik (JFM)* in years 1931–1939. He has written 56 reviews.

Marcinkiewicz papers besides the original and important results, contain a lot of ideas. They are still used today and continue to inspire mathematicians.

Antoni Zygmund has written about his pupil Marcinkiewicz ([Zy64], p. 1):

> *I was one of his professor at the University in Wilno; I introduced him to mathematical research and interested him in problems with which I was then concerned. Later on we collaborated and wrote several joint papers; but his scientific development was so rapid and the originality of his ideas so great that in certain parts of my own field of work I may only consider myself as his pupil.*

Orlicz asserted that Marcinkiewicz:

> *is probably the only mathematician with whom you can speak about everything. He was a remarkably quick learner.*

[12]Borge Christian Jessen (born 19 June 1907 in Copenhagen – died 20 March 1993 in Copenhagen).

Marcinkiewicz was one of the most eminent figures in Polish mathematics and together with Stanisław Zaremba, Leon Lichtenstein, Juliusz Schauder and Antoni Zygmund, the most prominent in classical analysis. Zygmund considered him his best pupil, although he had many students in Poland and the USA. We must also remember that it is because of the Master, which was Zygmund, Marcinkiewicz also became a known mathematician.

Photo 14. Wilno, 4 March 1936. Józef Marcinkiewicz (left) and Antoni Zygmund

Paul Nevai in the text of Paul Turán informed ([Tu55], p. 3):

> *Zygmund told me that Marcinkiewicz was the strongest mathematician he ever met – I wonder if I am making this up or he told this to others as well.*

Alberto P. Calderón described ([Ca83], p. xiv) ([Tu55], p. 3):

> *Marcinkiewicz, whose name is familiar to everyone interested in functional analysis and Fourier series, was an extraordinary mathematician. His collaboration with Zygmund lasted almost ten years and produced a number of important results.*

Cora Sadosky, in added information on A. Zygmund and J. Marcinkiewicz, conclude ([Sa01], p. 6):

> *Marcinkiewicz, did became a first-rank mathematician even if he died at 30. Marcinkiewicz is recognized today largely because Zygmund survived the war and became his champion.*

Marcinkiewicz name in mathematics appeared e.g. in connection to the following: the Marcinkiewicz interpolation theorem, Marcinkiewicz spaces, the Marcinkiewicz integral and Marcinkiewicz function, the Marcinkiewcz–Zygmund inequalities, the Marcinkiewcz–Zygmund law of large numbers, the Marcinkiewicz multiplier theorem, the Jessen–Marcinkiewicz–Zygmund strong differentiation theorem, Marcinkiewcz–Zygmund vector-valued inequalities, the Grünwald–Marcinkiewicz interpolation theorem, the Marcinkiewicz–Salem conjecture, the Marcinkiewicz test for pointwise convergence of Fourier series, the Marcinkiewicz theorem on the Haar system, the Marcinkiewicz theorem on universal primitive function and the Marcinkiewicz theorem on the Perron integral.

A description of Marcinkiewicz achievements was written in Polish by Antoni Zygmund [Zy60] in 1960 and then translated into English and published in the Collected Papers [Zy64] (pages 1–33). Achievements of Marcinkiewicz in analysis were described in Japanese by Satoru Igari [Ig05] (the English translation was published three years later in [Ig08]). We note, moreover, that Philip Holgate delivered on 25 February 1989 a lecture on *Independent functions: probability and analysis in Poland between the wars*, which was published in 1997 (see [Ho97]), and in the third part of this work some important achievements of Marcinkiewicz and Zygmund were discussed.

Also, descriptions of Zygmund achievements have been written by Fefferman, Kahane and Stein [FKS76] and by Stein [St83], which, of course, also contain the discussion of the joint results of Marcinkiewicz and Zygmund.

The article by Zygmund [Zy60] is obviously the best source of Marcinkiewicz mathematics, however, I have adopted an alternative order of presentation of Marcinkiewicz's results. My order follows appearence of Marcinkiewicz's results in text-books and monographs. This difference is also apparent from the fact that after 50 years, certain sections of mathematics became more popular than others. This is why the first are results in mathematical analysis (in fact in functional analysis), then in probability theory and real analysis to be finished with the classical Fourier series and general orthogonal series, and approximation theory, though Marcinkiewicz began to write papers on Fourier series and approximation theory. This work ends with remarks on some less cited or quoted papers of Marcinkiewicz.

4.1. Functional Analysis. In this section some results are presented that made Marcinkiewicz's name famous in the most spectacular way. These include the Marcinkiewicz interpolation theorem and two types of Marcinkiewicz spaces, as well as Marcinkiewicz–Zygmund vector-valued estimates of operators.

4.1.1. *Marcinkiewicz interpolation theorem* (*1939*). Consider two classical operators:

(a) The *Hardy operator* H is defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt, x \in I,$$

where $I = (0,a), 0 < a \leq \infty$. The operator H is not bounded from $L^1(I)$ to $L^1(I)$ (for example, $f_0(x) = \frac{1}{x\ln^2 x}\chi_{(0,1/2)} \in L^1(0,1)$, but $Hf_0(x) = -1/(x\ln x)$ on $(0,1/2)$ so that $Hf_0 \notin L^1(0,1)$), but it is bounded from $L^1(I)$ to weak-$L^1(I)$ and is bounded from $L^\infty(I)$ to $L^\infty(I)$.

(b) The *maximal operator* M is defined by

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|}\int_I |f(t)|\,dt, I = (a,b) \subset [0,1].$$

M is not bounded from L^1 to L^1, but is bounded from $L^1(I)$ to weak-$L^1(I)$ and it is bounded from L^∞ to L^∞.

In fact, there are several examples of such type of operators. Marcinkiewicz knowing the Riesz–Thorin interpolation theorem (1926, 1938) and the Kolmogorov result (1925) that the conjugate operator is of weak type $(1,1)$ tried to prove theorems not only for the scale of L^p-spaces. A particular version of the Marcinkiewicz interpolation theorem (1939) has the form: *If a linear or sublinear mapping T is of weak type $(1,1)$ and strong type (∞,∞), that is, satisfies the estimates*

$$\lambda\, m(\{x \in I : |Tf(x)| > \lambda\}) \le A \int_I |f(x)|\, dx \quad \forall \lambda > 0 \tag{1}$$

$$\operatorname*{ess\,sup}_{x\in I} |Tf(x)| \le B \operatorname*{ess\,sup}_{x\in I} |f(x)|, \tag{2}$$

then it is of strong type (p,p) for $1 < p \le \infty$, i.e., we have the estimate

$$\int_I |Tf(x)|^p\, dx \le C_{A,B} \int_I |f(x)|^p\, dx \tag{3}$$

with

$$C_{A,B}^{1/p} \le \frac{2}{(p-1)^{1/p}}\, A^{1/p}\, B^{1-1/p}. \tag{4}$$

To understand better the importance of the Marcinkiewicz interpolation theorem let us define the so-called *weak-L^p spaces*. The Lebesgue spaces L^p on the measure space (Ω, Σ, μ) were known for $\Omega = [a,b]$ or $\Omega = \mathbb{R}^n$ already in the thirties for F. Riesz. Investigating the boundedness of operators Marcinkiewicz needed larger spaces in the target, the so-called *weak-L^p spaces* denoted by $L^{p,\infty}$ $(1 \le p < \infty)$ and now called *Marcinkiewicz spaces* given by one of two quasi-norms:

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) = \sup_{\lambda>0} \lambda\, \mu(\{x \in \Omega : |f(x)| > \lambda\})^{1/p}.$$

Note that these spaces are larger than L^p, since for any $\lambda > 0$

$$\int_\Omega |f(x)|^p d\mu \ge \int_{\{x\in\Omega : |f(x)|>\lambda\}} |f(x)|^p d\mu \ge \int_{\{x\in\Omega : |f(x)|>\lambda\}} \lambda^p d\mu$$

$$= \lambda^p \mu(\{x \in \Omega : |f(x)| > \lambda\}),$$

that is,

$$\|f\|_{p,\infty} = \sup_{\lambda>0} \lambda\, \mu(\{x \in \Omega : |f(x)| > \lambda\})^{1/p} \le \left(\int_\Omega |f(x)|^p d\mu\right)^{1/p} = \|f\|_p.$$

Marcinkiewicz published in 1939 two-pages paper [M39h] and formulated three theorems, containing the one presented below, without proofs. Marcinkiewicz have sent a letter, including the proof of the main theorem for $p_0 = q_0 = 1$ and $p_1 = q_1 = 2$, to Zygmund who after the war reconstructed all the proofs and published them in 1956 (see [Zy56]). This is the reason why Marcinkiewicz interpolation theorem (1939) is sometimes called Marcinkiewicz–Zygmund interpolation theorem. Note that in 1953 Zygmund presented all the proofs at his Chicago seminar informing that he was only developing Marcinkiewicz's ideas (cf. [Pe02], p. 46). A proof was also given by the PhD students of Zygmund: Mischa Cotlar for $p_0 = q_0$ and $p_1 = q_1$ (PhD 1953, published in 1956 in [C56]) and William J. Riordan for $1 \le p_i \le q_i \le \infty, i = 0, 1$, (PhD 1955, unpublished).

Theorem 1 (Marcinkiewicz 1939, Zygmund 1956). *Let* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *and for* $0 < \theta < 1$ *define* p, q *by the equalities*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{5}$$

If $p_0 \leq q_0$ *and* $p_1 \leq q_1$ (lower triangle) *with* $q_0 \neq q_1$, *then from boundedness of any linear or sublinear operator* $T\colon L^{p_0} \to L^{q_0,\infty}$ *and* $T\colon L^{p_1} \to L^{q_1,\infty}$, [*i.e.* T *is of weak type* (p_0, q_0) *and of weak type* (p_1, q_1)] *follows boundedness* $T\colon L^p \to L^q$ [*that is,* T *is of strong type* (p, q)] *and*

$$\|T\|_{L^p \to L^q} \leq C \|T\|^{1-\theta}_{L^{p_0} \to L^{q_0,\infty}} \|T\|^{\theta}_{L^{p_1} \to L^{q_1,\infty}}, \tag{6}$$

where

$$C = C(\theta, p_0, p_1, q_0, q_1) \leq \frac{p_0^{(1-\theta)/p_0} p_1^{\theta/p_1}}{p^{1/p} q^{1/q}} \frac{2}{[|\frac{1}{q_1} - \frac{1}{q_0}| \theta(1-\theta)]^{1/q}}$$
$$\leq \frac{2}{[|\frac{1}{q_1} - \frac{1}{q_0}| \theta(1-\theta)]^{1/q}}$$

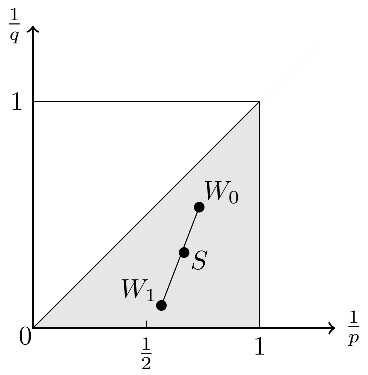


Fig. 1. Marcinkiewicz interpolation theorem is true in lower triangle: $W_0 = (\frac{1}{p_0}, \frac{1}{q_0})$, $W_1 = (\frac{1}{p_1}, \frac{1}{q_1}), S = (\frac{1}{p}, \frac{1}{q})$ and $q_0 \neq q_1$ (except horizontal segments)

Particular cases of Theorem 1 have the following form A and B:

A (Marcinkiewicz interpolation theorem (diagonal case)). *If* $1 \leq p_0 < p_1 \leq \infty$ *and* T *is an arbitrary linear or sublinear operator of weak type* (p_0, p_0) *and of weak type* (p_1, p_1) *for* $p_1 < \infty$ [*while for* $p_1 = \infty$ *we assume that it is of strong type* (∞, ∞)], *that is, bounded* $T\colon L^{p_0} \to L^{p_0,\infty}$ *and* $T\colon L^{p_1} \to L^{p_1,\infty}$, *then it is of strong type (p, p), i.e, bounded* $T\colon L^p \to L^p$ *and*

$$\|T\|_{L^p \to L^p} \leq 2 \left(\frac{p}{p - p_0} - \frac{p}{p_1 - p} \right)^{1/p} \|T\|^{1-\theta}_{L^{p_0} \to L^{p_0,\infty}} \|T\|^{\theta}_{L^{p_1} \to L^{p_1,\infty}}. \tag{7}$$

Remark 1. If the operator T in the above theorem is of weak type (p_0, p_0) and strong type (p_1, p_1), then of course it will be of strong type (p, p), but we obtain then better (than (7)) estimate on the norm, namely the following:

$$\|T\|_{L^p \to L^p} \le 2 \left(\frac{p}{p-p_0} \right)^{1/p} \|T\|^{1-\theta}_{L^{p_0} \to L^{p_0,\infty}} \|T\|^{\theta}_{L^{p_1} \to L^{p_1}}. \tag{8}$$

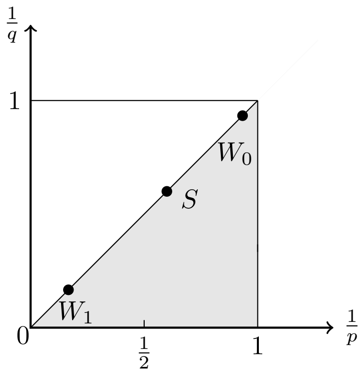


Fig. 2. Marcinkiewicz interpolation theorem (diagonal case):
$W_0 = (\frac{1}{p_0}, \frac{1}{p_0}), W_1 = (\frac{1}{p_1}, \frac{1}{p_1}), S = (\frac{1}{p}, \frac{1}{p})$.

B (Little Marcinkiewicz interpolation theorem). *If $1 < p \le \infty$ and T is an arbitrary linear or sublinear operator of weak type $(1,1)$ and strong type (∞, ∞), that is, bounded $T\colon L^1 \to L^{1,\infty}$ and $T\colon L^\infty \to L^\infty$, then it is of strong type (p, p), i.e., bounded $T\colon L^p \to L^p$ and*

$$\|T\|_{L^p \to L^p} \le \frac{2}{(p-1)^{1/p}} \|T\|^{1/p}_{L^1 \to L^{1,\infty}} \|T\|^{1-1/p}_{L^\infty \to L^\infty}. \tag{9}$$

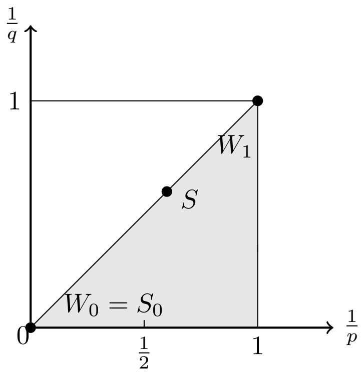


Fig. 3. Little Marcinkiewicz interpolation theorem:
$W_0 = S_0 = (0,0), W_1 = (1,1), S = (\frac{1}{p}, \frac{1}{p})$.

A natural question appears, namely if the reverse theorem is true, but there are linear and sublinear bounded operators in L^p for all $1 < p \le \infty$, which are not of weak type $(1,1)$. For example, such an operator is the two- or more dimensional strong maximal operator (averaging and supremum are taken over rectangles) as sublinear operator, and as a linear operator we can take arbitrary linear operator majorized by this sublinear operator. In connection to the Marcinkiewicz interpolation theorem we formulate several remarks:

1. Marcinkiewicz was probably the first who used the word "interpolation of operators". Riesz and Thorin spoke on "convexity theorems" (see also Peetre [Pe02], p. 39 and Horwath [Ho09], p. 618).

2. It is not true, as some authors write, that Marcinkiewicz obtained in his short paper the result only in the diagonal case. Marcinkiewicz had the theorem in the general case and equation (5) was written by the formula $\frac{q-q_0}{q_1-q} = \frac{q_0 p_1}{p_0 q_1} \cdot \frac{p-p_0}{p_1-p}$. Moreover, Marcinkiewicz's second theorem was formulated even for Orlicz spaces in the thesis (in this case it was indeed the diagonal case): if a linear or sublinear operator is bounded $T\colon L^{p_0} \to L^{p_0}$ and $T\colon L^{p_1} \to L^{p_1}$, and a function $\varphi\colon [0,\infty) \to [0,\infty)$ is continuous, increasing, vanishing at zero and satisfies three conditions $\varphi(2u) = O(\varphi(u))$, $\int_1^u t^{-p_0-1}\varphi(t)dt = o(u^{-p_0}\varphi(u))$ and $\int_u^\infty t^{p_1-1}\varphi(t)dt = o(u^{-p_1}\varphi(u))$ as $u \to \infty$, then for f such that $\varphi(|f|) \in L^1[0,1]$ we obtain $\int_0^1 \varphi(|Tf(x)|)dx \le C\int_0^1 \varphi(|f(x)|)dx + C$, where C is independent of f (see Zygmund [Zy56], Theorem 2 and [Zy59], XII. Theorem 4.22).

3. Marcinkiewicz interpolation theorem remains true for pointwise quasi-additive operators, that is, if there exists a constant $\gamma \ge 1$ such that for any measurable functions f, g we have the following inequality μ-almost everywhere on Ω:

$$|T(f+g)(x)| \le \gamma\left(|Tf(x)| + |Tg(x)|\right).$$

4. Evidently the constant $C(\theta, p_0, p_1, q_0, q_1) \to \infty$ as $\theta \to 0^+$ or $\theta \to 1^-$.

5. Marcinkiewicz interpolation theorem is true, if $p \le q$ and $q_0 \ne q_1$, that is, one point can be in the upper triangle but the point S must appear in the lower triangle. Proof of this theorem was given by: Calderón (1963), Lions–Peetre (1964), O'Neil (1964), Hunt (1964) and Krée (1967). Berenstein–Cotlar–Kerzman–Krée (1967) proved that if for the segment W_0W_1 Marcinkiewicz interpolation theorem is true, then it is also true for another segment obtained from the rotation of W_0W_1 around the point S (except horizontal and vertical segments).

6. Marcinkiewicz interpolation theorem is NOT true in the upper triangle, that is, if $p_0 > q_0$ and $p_1 > q_1$. A counter-example is due to Hunt 1964.

7. E. Stein and G. Weiss (1959) generalized the Marcinkiewicz interpolation theorem replacing the spaces L^{p_i} in the domain of an operator by the smaller Lorentz spaces $L^{p_i,1}, i = 0,1$ (in fact, they have in the assumption of the Marcinkiewicz theorem only estimates for characteristic functions of measurable sets). Hence, if $p_i \le q_i, p_i \ne \infty, i = 0,1$ and $q_0 \ne q_1$, then the boundedness of $T\colon L^{p_0,1} \to L^{q_0,\infty}$ and $T\colon L^{p_1,1} \to L^{q_1,\infty}$ implies the boundedness $T\colon L^p \to L^q$.

8. Using reiteration theorems for the real method of interpolation we are getting a *generalized Marcinkiewicz interpolation theorem*: *If a quasi-linear operator* $T\colon L^{p_0,1} \to L^{q_0,\infty}$ *and* $T\colon L^{p_1,1} \to L^{q_1,\infty}$ *is bounded, then* $T\colon L^{p,r} \to L^{q,r}$ *is bounded for any* $1 \le r \le \infty$. *In particular,* $T\colon L^p \to L^{q,p}$ *is bounded.*

If we have, as in Marcinkiewicz interpolation theorem, $p_0 \le q_0$ and $p_1 \le q_1$, then $L^{q,p} \hookrightarrow L^q$.

9. A very important progression to a generalization of the Marcinkiewicz interpolation

theorem was done by Calderón (1966), who found the maximal operator in the sense that if an operator $T\colon L^{p_0,1} \to L^{q_0,\infty}$ and $T\colon L^{p_1,1} \to L^{q_1,\infty}$ is bounded, then $(Tf)^*(t) \leq CS_\sigma(f^*)(t)$ for all $t > 0$, where S_σ is a maximal Calderón operator:

$$\begin{aligned} S_\sigma f(t) &= \int_0^\infty f(s) \min\{\frac{s^{1/p_0}}{t^{1/q_0}}, \frac{s^{1/p_1}}{t^{1/q_1}}\}\frac{ds}{s} \\ &= t^{-1/q_0}\int_0^{t^m} s^{1/p_0-1}f(s)ds + t^{-1/q_1}\int_{t^m}^\infty s^{1/p_1-1}f(s)ds, \end{aligned}$$

with $m = (1/q_0 - 1/q_1)/(1/p_0 - 1/p_1)$. To get Marcinkiewicz interpolation theorem it is enough to investigate boundedness of the last two operators of Hardy type.

10. Marcinkiewicz interpolation theorem was proved for symmetric spaces by Boyd (1967 $\Rightarrow$, 1969 $\Leftrightarrow$): if $1 \leq p_0 < p_1 < \infty$, E is a symmetric space with the Fatou property of the norm on either $I = [0,1]$ or $I = [0,\infty)$ and linear operator operator $T\colon L^{p_0,1} \to L^{p_0,\infty}$ and $T\colon L^{p_1,1} \to L^{p_1,\infty}$ is bounded, then it yields that $T\colon E \to E$ is bounded if and only if $1/p_1 < \alpha_E \leq \beta_E < 1/p_0$, where numbers α_E, β_E are so-called Boyd indices of the space E defined by

$$\alpha_E = \lim_{a\to 0^+} \frac{\ln \|\sigma_a\|_{E\to E}}{\ln a}, \quad \beta_E = \lim_{a\to\infty} \frac{\ln \|\sigma_a\|_{E\to E}}{\ln a}$$

and $\sigma_a f(x) = f(x/a)\chi_I(x/a)$. Krein–Petunin–Semenov (1977) proved that Boyd's theorem is true for arbitrary symmetric spaces (even without the Fatou property of the norm). In particular, E is an interpolation space between L_{p_0} and L_{p_1}. If $p_1 = \infty$, a one-sided estimate for a symmetric space E, $\beta_E < 1/p_0, 1 \leq p_0 < \infty$, implies that E is an interpolation space between L_{p_0} and L_∞ (see Maligranda [Mal81], Thm 4.6, where it is proved even for Lipschitz operators). Moreover, Astashkin and Maligranda [AM04] proved the following one-sided Boyd theorem: if a symmetric space E has either the Fatou property or it is separable and $\alpha_E > 1/p_1, 1 < p_1 < \infty$, then E is an interpolation space between L_1 and L_{p_1}.

11. Marcinkiewicz interpolation theorem is not true for bilinear operators without additional assumptions. In fact, Strichartz (1969) proved the following: the operator

$$S(f,g)(x) = \int_0^\infty f(xt)g(t)dt$$

is bounded $S\colon L^1 \times L^\infty \to L^{1,\infty}$ and $S\colon L^2 \times L^2 \to L^{2,\infty}$, but it is not bounded $S\colon L^p \times L^{p'} \to L^p$, and Maligranda (1989) proved that the operator

$$T(f,g)(x) = \int_0^1\int_0^1 f(s)g(t)\min(\frac{1}{st}, \frac{1}{x})dsdt$$

is bounded $T\colon L^1 \times L^1 \to L^{1,\infty}$ and $T\colon L^2 \times L^2 \to L^{2,\infty}$, but it is not bounded $T\colon L^p \times L^p \to L^p$ for $1 < p \leq 2$.

J. L. Lions and J. Peetre (1964) proved that for the real method of interpolation we have the following interpolation theorem for bilinear operators: if a bilinear operator $T\colon L^{p_0} \times L^{q_0} \to L^{r_0,\infty}$ and $T\colon L^{p_1} \times L^{q_1} \to L^{r_1,\infty}$ is bounded, then $T\colon L^p \times L^q \to L^r$ is bounded, if besides the natural interpolation equality

$$(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}) = (1-\theta)(\frac{1}{p_0}, \frac{1}{q_0}, \frac{1}{r_0}) + \theta(\frac{1}{p_1}, \frac{1}{q_1}, \frac{1}{r_1})$$

we have also $1/r \leq 1/p + 1/q - 1$. More theorems of this type can be found in papers by Sharpley (1977), Zafran (1978), Janson (1988) and Grafakos–Kalton (2001).

12. Marcinkiewicz interpolation theorem for spaces of sequences (and some of its analogues) were given by Sargent in 1961.

13. In 1972 Yoram Sagher (cf. [Sa72], p. 172 and [Sag72], p. 240) introduced the notion of a *Marcinkiewicz quasi-cone*. If (A_0, A_1) is a pair of quasi-normed spaces, then a subset Q of $A_0 + A_1$ is called a quasi-cone if $Q + Q \subset Q$. Q is a cone if we also have $\lambda Q \subset Q$ for all $\lambda > 0$. A quasi-cone Q is called *Marcinkiewicz quasi-cone* in (A_0, A_1) if

$$(A_0 \cap Q, A_1 \cap Q)_{\theta,p} = (A_0, A_1)_{\theta,p} \cap Q \quad \text{for all } 0 < \theta < 1, 0 < p \leq \infty,$$

where $(\cdot,\cdot)_{\theta,p}$ means the real K-method of interpolation of Lions-Peetre. For example, $Q = \{(x_k)_{k=1}^{\infty} : x_k \downarrow 0\}$ is a Marcinkiewicz quasi-cone in (l^p, l^∞).

Marcinkiewicz interpolation theorem is cited in several classical books on analysis, harmonic analysis and interpolation theory as, for example, in the following 55 books (in chronological order):

[Co59] M. Cotlar, *Condiciones de Continuidad de Operadores Potenciales y de Hilbert*, Cursos y Seminarios de Matemática, Fasc. 2, Universidad Nacional de Buenos Aires, Buenos Aires 1959.

[Zy59] A. Zygmund, *Trigonometric Series, Vol. I, II.*, Cambridge Univ. Press, Cambridge 1959 [Chapter XII, Part 4. Marcinkiewicz's theorem on the interpolation of operators, pp. 111–120]; Russian transl., Izdat. Mir, Moscow 1965.

[DS63] N. Dunford and J. T. Schwartz, *Linear Operators. Part II: Spectral Theory. Self Adjoint Operators in Hilbert Space*, John Wiley & Sons, New York-London 1963 [important interpolation theorem of Marcinkiewicz, pp. 1166–1168].

[KZPS66] M. A. Krasnosel'skiĭ, P. P. Zabreĭko, E. I. Pustyl'nik and P. E. Sobolevskiĭ, *Integral Operators in Spaces of Summable Functions*, Nauka, Moscow 1966 (Russian) [2. 7. The Marcinkiewicz interpolation theorem, pp. 47–55]; English transl. Nordhoff, Leyden 1976 [2. 7. The Marcinkiewicz interpolation theorem, pp. 39–47].

[St70a] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton 1970 [Chapter I, part 4. An interpolation for L^p and in the text the Marcinkiewicz interpolation theorem, pp. 20–22; Appendix B: Marcinkiewicz interpolation theorem].

[St70b] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Princeton Univ. Press, Princeton and University of Tokyo Press, Tokyo 1970 [Marcinkiewicz interpolation theorem, pp. 92–93].

[Ok71] G. O. Okikiolu, *Aspects of the Theory of Bounded Integral Operators in L^p-Spaces*, Academic Press, London-New York 1971 [Part 5.2. Distribution functions and the Marcinkiewicz–Zygmund interpolation theorem, pp. 234–245].

[SW71] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton 1971 [V.2. Marcinkiewicz interpolation theorem, pp. 183–200].

[CC74] M. Cotlar and R. Cignoli, *An Introduction to Functional Analysis*, North-Holland, Amsterdam-London and American Elsevier, New York 1974 [3.1.3. Interpolation theorem of Marcinkiewicz for weak type (p, p), pp. 387–388; 3.1.4. Interpolation theorem of Marcinkiewicz–Zygmund, pp. 388–389].

[RS75] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, New York-London 1975 [IX.18. Marcinkiewicz interpolation theorem].

[BL76] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin-New York 1976 [Part 1.3. The Marcinkiewicz theorem, pp. 6–11; 1.4. An application of the Marcinkiewicz theorem, pp. 11–12]; Russian transl. Mir, Moscow 1980.

[EG77] R. E. Edwards and G. I. Gaudry, *Littlewood–Paley and Multiplier Theory*, Springer, Berlin-New York 1977 [A. 2. Marcinkiewicz interpolation theorems, pp. 179–183].

[KJF77] A. Kufner, O. John and S. Fučik, *Function Spaces*, Academia, Prague 1977 [Marcinkiewicz's theorem, pp. 106].

[BIN78] O. V. Besov, V. P. Ilin and S. M. Nikolskiĭ, *Integral Representations of Functions and Imbedding Theorems, Vol. I*, Wiley, New York 1978 [3.2. Marcinkiewicz's theorem, pp. 50–53].

[Tr78] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam-New York 1978 [Theorem 3, pp. 136–137; Russian transl. Mir, Moscow 1980 [Theorem 3, pp. 159–161].

[Sa79] C. Sadosky, *Interpolation of Operators and Singular Integrals. An Introduction to Harmonic Analysis*, Marcel Dekker, New York 1979 [4.4. The Marcinkiewicz interpolation theorem: diagonal case, pp. 169–179; 4.5. The Marcinkiewicz interpolation theorem: general case, pp. 179–190].

[LT79] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, II. Function Spaces*, Springer, Berlin-New York 1979 [Theorem 2.b15 (The Marcinkiewicz interpolation theorem), pp. 149–150].

[Ru80] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer, New York-Berlin 1980 [5.7. Appendix: Marcinkiewicz interpolation, pp. 88–90].

[Gu81] M. de Guzmán, *Real Variable Methods in Fourier Analysis*, North-Holland, Amsterdam-New York 1981 [3.4.B. The Marcinkiewicz theorem, pp. 55].

[NP81] C. Niculescu and N. Popa, *Elements of the Theory of Banach Spaces*, Editura Acad. Rep. Soc. Romania, Bucharest 1981 (Romanian) [2.8.2. Marcinkiewicz interpolation theorem].

[Ed82] R. E. Edwards, *Fourier Series. Vol. 2. A Modern Introduction*, 2nd ed., Springer, New York-Berlin 1982 [13.8. Marcinkiewicz interpolation theorem].

[JM82] O. G. Jorsboe and L. Mejlbro, *The Carleson–Hunt Theorem on Fourier Series*, Lecture Notes in Mathematics 911, Springer, Berlin-New York 1982 [Theorem 1.9 – theorem due to Marcinkiewicz, pp. 8–10].

[KPS82] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978 (Russian); English transl. in AMS, Providence 1982 [Theorem 6.1 and 6.1′ are extensions of Marcinkiewicz' theorem, pp. 129–130, 132–133 and 350].

[KS84] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow 1984 (Russian) [Appendix 1.2. Marcinkiewicz interpolation theorem, pp. 442–443]; English transl. AMS, Providence 1989 [Appendix 2.1. Marcinkiewicz interpolation theorem, pp. 390–392]; Second Russian Edition 1999.

[GR85] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam 1985 [2.11. Marcinkiewicz interpolation theorem, pp. 148–150].

[To86] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, Orlando 1986 [Chapter IV, Part 4. The Marcinkiewicz interpolation theorem, pp. 86–91].

[BS88] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston 1988 [Part 4.4. The Marcinkiewicz interpolation theorem, pp. 216–230].

[Bu89] V. I. Burenkov, *Functional Spaces. Basic Integral Inequalities Connected with* L_p*-Spaces*, Izd. Univ. Druzhby Narodov, Moscow 1989 (Russian) [Part 5.3. Marcinkiewicz theorem, pp. 78–86].

[Zi89] W. P. Ziemer, *Weakly Differentiable Functions*, Springer, New York 1989 [4.7.1. Marcinkiewicz interpolation theorem, p. 199].

[BK91] Yu. A. Brudnyĭ and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces I*, North-Holland, Amsterdam 1991 [Part 1.10. The Marcinkiewicz theorem, pp. 66–83].

[RR91] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York 1991 [Marcinkiewicz's interpolation theorem for certain Orlicz spaces, Theorem 13, pp. 247–252].

[Me92] Y. Meyer, *Wavelets and Operators*, Cambridge Univ. Press, Cambridge 1992 [Marcinkiewicz's theorem, p. 166].

[So93] C. D. Sogge, *Fourier Integrals in Classical Analysis*, Cambridge Univ. Press, Cambridge 1993 [Marcinkiewicz interpolation theorem, pp. 12–14].

[St93] D. W. Stroock, *Probability Theory. An Analytic View*, Cambridge Univ. Press, Cambridge 1993 [6.2.19. Marcinkiewicz theorem, pp. 325–326].

[Ba95] R. F. Bass, *Probabilistic Techniques in Analysis*, Springer, New York 1995 [Marcinkiewicz interpolation theorem, pp. 31–32 and 304–305].

[Wo97] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, Cambridge Univ. Press, Cambridge 1997 [6.7. Marcinkiewicz interpolation theorem, pp. 145–146]; Polish transl. *Teoria falek*, PWN, Warszawa 2000 [7.7. Marcinkiewicz theorem, pp. 143–144].

[CW98] Y.-Z. Chen and L.-C. Wu, *Second Order Elliptic Equations and Elliptic Systems*, AMS, Providence 1998 [3.1. The Marcinkiewicz interpolation theorem, pp. 37–39].

[Ko98] P. Koosis, *Introduction to H_p Spaces*, Cambridge Univ. Press, Cambridge 1998 [2. Proof of M. Riesz' theorem by Marcinkiewicz interpolation, p. 94].

[Fo99] G. B. Folland, *Real Analysis. Modern Techniques and their Applications*, 2nd ed., John Wiley & Sons, New York 1999 [Theorem 6.28. The Marcinkiewicz interpolation theorem, pp. 203–208].

[St00] V. D. Stepanov, *Some Topics in the Theory of Integral Convolution Operators*, Dalnauka, Vladivostok 2000 (Russian) [Marcinkiewicz theorem, pp. 92–93].

[Du01] Z. Duoandikoetxea, *Fourier Analysis*, AMS, Providence 2001 [2.3. The Marcinkiewicz interpolation theorem, pp. 28–30].

[GT01] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin 2001 [9.3. The Marcinkiewicz interpolation theorem, pp. 227–230].

[Ar02] J. Arias de Reyna, *Pointwise Convergence of Fourier Series*, Lecture Notes in Math. 1785, Springer, Berlin 2002 [11.4. The Marcinkiewicz interpolation theorem, pp. 134–137].

[Di02] E. DiBenedetto, *Real Analysis*, Birkhäuser, Boston 2002 [VIII.9. The Marcinkiewicz interpolation theorem, pp. 390–394].

[Ni02] N. K. Nikolski, *Operators, Functions, and Systems: an Easy Reading, Vol. 1. Hardy, Hankel, and Toeplitz*, AMS, Providence 2002 [Marcinkiewicz interpolation theorem, pp. 121–122].

[Pi02] M. A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, Brooks/Cole, Pacific Grove, CA 2002 [3.6. The Marcinkiewicz interpolation theorem, pp. 206–208]; reprint AMS, Providence 2009.

[Ka03] S. Kantorovitz, *Introduction to Modern Analysis*, Oxford Univ. Press, Oxford 2003 [5.8. Operators between Lebesgue spaces: Marcinkiewicz's interpolation theorem, pp. 145–150].

[Pa04] M. Pavlović, *Introduction to Function Spaces on the Disk*, Matematički Institut SANU, Belgrade 2004 [2.2. Weak L^p-spaces and Marcinkiewicz's theorem, pp. 22–25].

[Kn05] A. W. Knapp, *Basic Real Analysis*, Birkhäuser, Basel 2005 [IX.6. Marcinkiewicz interpolation theorem, pp. 427–436].

[Ta06] M. E. Taylor, *Measure Theory and Integration*, AMS, Providence 2006 [Appendix D. The Marcinkiewicz interpolation theorem, pp. 283–285].

[Ga07] D. J. H. Garling, *Inequalities: A Journey into Linear Analysis*, Cambridge Univ. Press, Cambridge 2007 [Part 10.1. The Marcinkiewicz interpolation theorem: I; pp. 154–156 and Part 10.5. The Marcinkiewicz interpolation theorem: II, pp. 162–165].

[Pi07] A. Pietsch, *History of Banach Spaces and Linear Operators*, Birkhäuser, Boston 2007 [The Marcinkiewicz interpolation theorem (1939), str. 433].

[AF08] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd ed., Academic Press, New York 2008 [2.58. The Marcinkiewicz interpolation theorem, pp. 55–58].

[Gr08] L. Grafakos, *Classical Fourier Analysis*, 2nd ed., Springer, New York 2008 [Part 1.3.1. Real method: The Marcinkiewicz interpolation theorem, pp. 31–34 and Part 1.4.4. The off-diagonal Marcinkiewicz interpolation theorem, pp. 55–63].

[LP09] F. Linares and G. Ponce, *Introduction to Nonlinear Dispersive Equations*, Springer, New York 2009 [2.2. Marcinkiewicz interpolation theorem (diagonal case), pp. 29–33].

A very long list of publications is devoted to all possible variants and generalizations of Marcinkiewicz interpolation theorem, whose part is as follows:

[AM04] S. V. Astashkin and L. Maligranda, *Interpolation between L_1 and L_p, $1 < p < \infty$*, Proc. Amer. Math. Soc. 132(2004), no. 10, 2929–2938.

[BR80] C. Bennett and K. Rudnick, *On Lorentz–Zygmund spaces*, Dissertationes Math. (Rozprawy Mat.) 175(1980), 1–67.

[BCKK67] C. A. Berenstein, M. Cotlar, N. Kerzman and P. Krée, *Some remarks on the Marcinkiewicz convexity theorem in the upper triangle*, Studia Math. 29(1967), 79–95.

[Bo69] D. W. Boyd, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. 21(1969), 1245–1254.

[BKS86] Yu. A. Brudnyĭ, S. G. Krein and E. M. Semenov, *Interpolation of linear operators*, Itogi Nauki i Tekhniki, Mat. Analiz 24, 3–163; English transl. in J. Soviet Math 42(1988), no. 6, 2009–2112 [Theorem of Marcinkiewicz, pp. 2017, 2058].

[Ca66] A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. 26(1966), 273–299.

[Ci98] A. Cianchi, *An optimal interpolation theorem of Marcinkiewicz type in Orlicz spaces*, J. Funct. Anal. 153(1998), no. 2, 357–381.

[Co56] M. Cotlar, *A general interpolation theorem for linear operations*, Rev. Mat. Cuyana 1(1955), 57–84 (1956).

[CB57] M. Cotlar and M. Bruschi, *On the convexity theorems of Riesz–Thorin and Marcinkiewicz*, Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie Segunda. Rev. 5(1956), 162–172 (1957) (Spanish).

[GK01] L. Grafakos and N. Kalton, *Some remarks on multilinear maps and interpolation*, Math. Ann. 319(2001), no. 1, 151–180.

[Hu64] R. A. Hunt, *An extension of the Marcinkiewicz interpolation theorem to Lorentz spaces*, Bull. Amer. Math. Soc. 70(1964), 803–807.

[Ig62] S. Igari, *An extension of the interpolation theorem of Marcinkiewicz*, Proc. Japan Acad. 38(1962), 731–734.

[Ig63] S. Igari, *An extension of the interpolation theorem of Marcinkiewicz. II*, Tohoku Math. J. (2) 15(1963), 343–358.

[Ja88] S. Janson, *On interpolation of multilinear operators*, in: "Function Spaces and Applications" (Lund, 1986), Lecture Notes in Math. 1302, Springer, Berlin 1988, 290–302.

[Kr67] P. Krée, *Interpolation d'espaces vectoriels qui ne sont ni normés, ni complets. Applications*, Ann. Inst. Fourier (Grenoble) 17(1967), fasc. 2, 137–174 (1968).

[LP64] J.-L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. 19(1964), 5–68.

[Mal81] L. Maligranda, *A generalization of the Shimogaki theorem*, Studia Math. 71(1981), no. 1, 69–83.

[Mal85] L. Maligranda, *Indices and interpolation*, Dissertationes Math. (Rozprawy Mat.) 234(1985), 1–54 [Chapter 6, Part B, Marcinkiewicz interpolation theorem in Orlicz spaces].

[Mal89] L. Maligranda, *On interpolation of nonlinear operators*, Comment. Math. Prace Mat. 28(1989), no. 2, 253–275.

[Ok65] E. T. Oklander, *Interpolación, espacios de Lorentz y teorema de Marcinkiewicz*, Cursos y Seminarios de Mat., Fasc. 20, Universidad de Buenos Aires, Buenos Aires 1965, 1–111.

[Ri71] N. M. Riviére, *Interpolation á la Marcinkiewicz*, Rev. Un. Mat. Argentina 25(1970/71), 363–377.

[Sa72] Y. Sagher, *An application of interpolation theory to Fourier series*, Studia Math. 41(1972), 169–181.

[Sag72] Y. Sagher, *Some remarks on interpolation of operators and Fourier coefficients*, Studia Math. 44(1972), 239–252.

[Sa61] W. L. C. Sargent, *Some analogues and extensions of Marcinkiewicz's interpolation theorem*, Proc. London Math. Soc. (3) 1(1961), 457–468.

[Sh 77] R. Sharpley, *Multilinear weak type interpolation of m n-tuples with applications*, Studia Math. 60(1977), no. 2, 179–194.

[SW59] E. M. Stein and G. Weiss, *An extension of a theorem of Marcinkiewicz and some of its applications*, J. Math. Mech. 8(1959), 263–284.

[St69] R. S. Strichartz, *A multilinear version of the Marcinkiewicz interpolation theorem*, Proc. Amer. Math. Soc. 21(1969), 441–444.

[To76] A. Torchinsky, *Interpolation of operations and Orlicz classes*, Studia Math. 59(1976), no. 2, 177–207.

[Za78] M. Zafran, *A multilinear interpolation theorem*, Studia Math. 6(1978), no. 2, 107–124.

[Zy56] A. Zygmund, *On a theorem of Marcinkiewicz concerning interpolation of operations*, J. Math. Pures Appl. (9) 35(1956), 223–248.

4.1.2. *Marcinkiewicz function and sequence spaces (1939).* Marcinkiewicz investigations led him to consider three types of spaces: two symmetric spaces and one space of another type, which will be considered in the next part. All are called now *Marcinkiewicz spaces.* Earlier we discussed about the weak-L^p space $L^{p,\infty}(1 \le p < \infty))$ or *Marcinkiewicz space* given by one of quasi-norms:

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) = \sup_{\lambda>0} \lambda\, \mu(\{x \in \Omega : |f(x)| > \lambda\})^{1/p}.$$

The above type of space can be easily generalized. Let $I = (0,1)$ or $I = (0,\infty)$ and let $\varphi \colon I \cup \{0\} \to [0,\infty)$ be an arbitrary concave function on I such that $\varphi(0) = 0$ (it is also possible to take as φ only quasi-concave function, that is, a function for which inequality $\varphi(s) \le \max(1, s/t)\varphi(t)$ is true for all $s,t \in I$). The *Marcinkiewicz function space* M^*_φ on I contains classes of all measurable functions generated by the quasi-norm

$$\|f\|^*_\varphi = \sup_{t\in I} \varphi(t)\, f^*(t) < \infty,$$

where f^* denotes the decreasing rearrangement of $|f|$.

Important is also another (smaller) *Marcinkiewicz function space* M_φ on I generated by the norm

$$\|f\|_\varphi = \sup_{t\in I} \varphi(t)\, f^{**}(t), \quad \text{where } f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds.$$

In the case when $\varphi(t) = t^{1/p}, 1 < p < \infty$ we have $M_\varphi^* = M_\varphi = L^{p,\infty}$, but for $\varphi(t) = t$ we get $M_\varphi^* = L^{1,\infty}$ (weak-L^1 space is a quasi-Banach space but not a Banach space, since the triangle inequality holds with constant 2 and $M_\varphi = L^1$).

Of course, $M_\varphi \subset M_\varphi^*$ and $\|f\|_\varphi^* \le \|f\|_\varphi$ for $f \in M_\varphi$. The Marcinkiewicz function space M_φ on I is a symmetric Banach space and for an arbitrary symmetric space X on I with the fundamental function $\varphi(t) := \|\chi_{(0,t)}\|_X$, M_φ is the smallest symmetric space containing X with the same fundamental function, i.e, $\|\chi_{(0,t)}\|_X = \|\chi_{(0,t)}\|_{M_\varphi} = \varphi(t)$ for any $t \in I$. This follows from the fact that $\int_0^t f^*(s)ds \le \frac{t}{\varphi(t)}\|f^*\chi_{(0,t)}\|_X$ for $f \in X$.

Let us recall that a symmetric space X on I is an ideal Banach space on I (the assumption $|f(t)| \le |g(t)|$ almost everywhere on $I, g \in X$ and f is measurable on I implies that $f \in X$ and $\|f\|_X \le \|g\|_X$) with the additional property that two arbitrary equimeasurable functions f and g, i.e. satisfying $m(\{x \in I : |f(x)| > \lambda\}) = m(\{x \in I : |g(x)| > \lambda\})$ for any $\lambda > 0$, with $f \in X$ and g measurable on I gives $g \in X$ and $\|f\|_X = \|g\|_X$. In particular, $\|f\|_X = \|f^*\|_X$.

Note that when we investigate the Marcinkiewicz spaces $L^{p,\infty}$ or operators with values in this space, then important is the so-called Kolmogorov–Cotlar equivalence (cf. García-Cuerva and Rubio de Francia [GR85], pp. 485–486): if $0 < q < p < \infty$ and $f \in L^{p,\infty}$, then

$$\|f\|_{p,\infty} \approx \sup_{A\subset I, 0<\mu(A)<\infty} \mu(A)^{1/p-1/q}\left(\int_A |f(x)|^q d\mu\right)^{1/q}. \tag{10}$$

Marcinkiewicz sequence spaces m_φ^* and m_φ are defined analogously by quasi-norms and norms

$$\|x\|_\varphi^* = \sup_{n\in\mathbb{N}} \varphi(n)\, x_n^* < \infty, \|x\|_\varphi = \sup_{n\in\mathbb{N}} \varphi(n)\,\frac{1}{n}\sum_{k=1}^n x_k^*,$$

where $\varphi\colon \mathbb{N} \to [0,\infty)$ satisfies $\varphi(k) \le \max(1, k/n)\varphi(n)$ for any $k, n \in \mathbb{N}$ and $(x_n{}^*)$ is a rearrangement of the sequence $(|x_n|)$ in decreasing order. These spaces are also symmetric sequence spaces.

Marcinkiewicz function and sequence spaces (symmetric) are now classical spaces and are still investigated (some examples are given below). They also appear naturally in the interpolation theory.

Marcinkiewicz symmetric spaces, as examples or objects of investigation of their structure, can be found e.g. in the following 5 monographs:

[KJF77] A. Kufner, O. John and S. Fučik, *Function Spaces*, Academia, Prague 1977 [Part 4.2. Marcinkiewicz spaces and their connection with the spaces $L_p^w(\Omega)$, pp. 209–212].

[PP80] Ju. I. Petunin and A. N. Plichko, *The Theory of the Characteristics of Subspaces and its Applications*, Vishcha Shkola, Kiev 1980 (Russian) [Marcinkiewicz spaces M_ψ and M_ψ^0, pp. 101–103 and 111–112].

[KPS82] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978 (Russian); English transl. in AMS, Providence 1982 [Chapter II, Part 5,

p. 107: Lorentz and Marcinkiewicz spaces and in 2. Marcinkiewicz spaces, pp. 112–118; Part 6.2. Operators from Lorentz spaces to Marcinkiewicz spaces, pp. 127–130].

[AZ90] J. Appell and P. P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Univ. Press, Cambridge 1990 [5.2. Lorentz and Marcinkiewicz spaces].

[BK91] Yu. A. Brudnyĭ and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces I*, North-Holland, Amsterdam 1991 [The Marcinkiewicz space $M(\varphi)$, p. 472].

The name *Marcinkiewicz space* appeared also in Encyclopedia of Mathematics:

[Kr90] S. G. Krein, *Marcinkiewicz space*, in: Encyclopaedia of Mathematics, Vol. 10, Kluwer, Dodrecht 1990, 93–94.

Papers having word *Marcinkiewicz space* in the title are for instance:

[AK08] M. D. Acosta and A. Kamińska, *Norm-attaining operators between Marcinkiewicz and Lorentz spaces*, Bull. Lond. Math. Soc. 40(2008), no. 4, 581–592.

[AS92] J. Appell and E. M. Semenov, *On the equivalence of the Lorentz and Marcinkiewicz norm on subsets of measurable functions*, J. Funct. Anal. 104(1992), no. 1, 47–53.

[AS95] J. Appell and E. M. Semenov, *On the truncation of functions in Lorentz and Marcinkiewicz spaces*, Rocky Mountain J. Math. 25(1995), no. 3, 857–866.

[As07] S. V. Astashkin, *On the normability of Marcinkiewicz classes*, Mat. Zametki 81(2007), no. 4, 483–489; English transl. in Math. Notes 81(2007), no. 3–4, 429–434.

[AL06] S. V. Astashkin and K. V. Lykov, *Extrapolation description of Lorentz and Marcinkiewicz spaces "close" to L_∞*, Sibirsk. Mat. Zh. 47(2006), no. 5, 974–992; English transl. in Siberian Math. J. 47(2006), no. 5, 79–812.

[AS07] S. V. Astashkin and F. A. Sukochev, *Banach–Saks property in Marcinkiewicz spaces*, J. Math. Anal. Appl. 336(2007), no. 2, 1231–1258.

[BL10] C. Boyd and S. Lassalle, *Geometry and analytic boundaries of Marcinkiewicz sequence spaces*, Q. J. Math. 61(2010), 183–197.

[CN85] M. Cwikel and P. Nilsson, *Interpolation of Marcinkiewicz spaces*, Math. Scand. 56(1985), no. 1, 29–42.

[KK05] N. J. Kalton and A. Kamińska, *Type and order convexity of Marcinkiewicz and Lorentz spaces and applications*, Glasg. Math. J. 47(2005), no. 1, 123–137.

[KL04] A. Kamińska and H. J. Lee, *M-ideal properties in Marcinkiewicz spaces*, Comment. Math. Prace Mat. 2004, Tomus specialis in Honorem Juliani Musielak, 123–144.

[KLL09] A. Kamińska, H. J. Lee and G. Lewicki, *Extreme and smooth points in Lorentz and Marcinkiewicz spaces with applications to contractive projections*, Rocky Mountain J. Math. 39(2009), no. 5, 1533–1572.

[KP08] A. Kamińska and A. M. Parrish, *Convexity and concavity constants in Lorentz and Marcinkiewicz spaces*, J. Math. Anal. Appl. 343(2008), no. 1, 337–351.

[KP10] A. Kamińska and A. M. Parrish, *Note on extreme points in Marcinkiewicz function spaces*, Banach J. Math. Anal. 4(2010), no. 1, 1–12.

[Lo75] G. Ja. Lozanovskiĭ, *Coordinate Marcinkiewicz spaces*, Optimizacija 17(34) (1975), 130–142 (Russian).

[Lo78] G. Ja. Lozanovskiĭ, *The representation of linear functionals in Marcinkiewicz spaces*, Izv. Vyssh. Uchebn. Zaved. Mat. 1978, no. 1(188), 43–53 (Russian).

[Me75] A. A. Mekler, *The Hardy–Littlewood property in Marcinkiewicz spaces*, Izv. Vysh. Uchebn. Zaved. Matematika 1975, no. 3(154), 104–106 (Russian).

[Mi78] M. Milman, *Embeddings of Lorentz–Marcinkiewicz spaces with mixed norms*, Anal. Math. 4(1978), no. 3, 215–223.

[Pu01] E. Pustylnik, *On some properties of generalized Marcinkiewicz spaces*, Studia Math. 144(2001), no. 3, 227–243.

[Se09] A. A. Sedaev, *Singular symmetric functionals and stabilizing subsets of the Marcinkiewicz space*, Izv. Vyssh. Uchebn. Zaved. Mat. 2009, no. 12, 90–94; English transl. in Russian Math. (Iz. VUZ) 53(2009), no. 12, 77–80.

[To78] E. V. Tokarev, *Quotient spaces of Marcinkiewicz spaces*, Sibirsk. Mat. Zh. 19(1978), no. 3, 704–707; English transl. in Siberian Math. J. 19(1978), no. 3, 498–500 (1979).

[To84] E. V. Tokarev, *Quotient spaces of Banach lattices and Marcinkiewicz spaces*, Sibirsk. Mat. Zh. 25(1984), no. 2, 205–212; English transl. in Siberian Math. J. 25(1984), no. 2, 332–338.

4.1.3. *Marcinkiewicz $M^p(\mathbb{R})$ spaces (1939).* The third type of Marcinkiewicz function spaces (sometimes also called Besicovitch–Marcinkiewicz spaces) consists of functions defined on $\mathbb{R}$ satisfying certain condition, more precisely, if $1 \le p < \infty$, then the *Marcinkiewicz space $M^p = M^p(\mathbb{R})$* contains all measurable real (or complex) functions $f\colon \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) locally p-integrable on $\mathbb{R}$ and such that the semi-norm

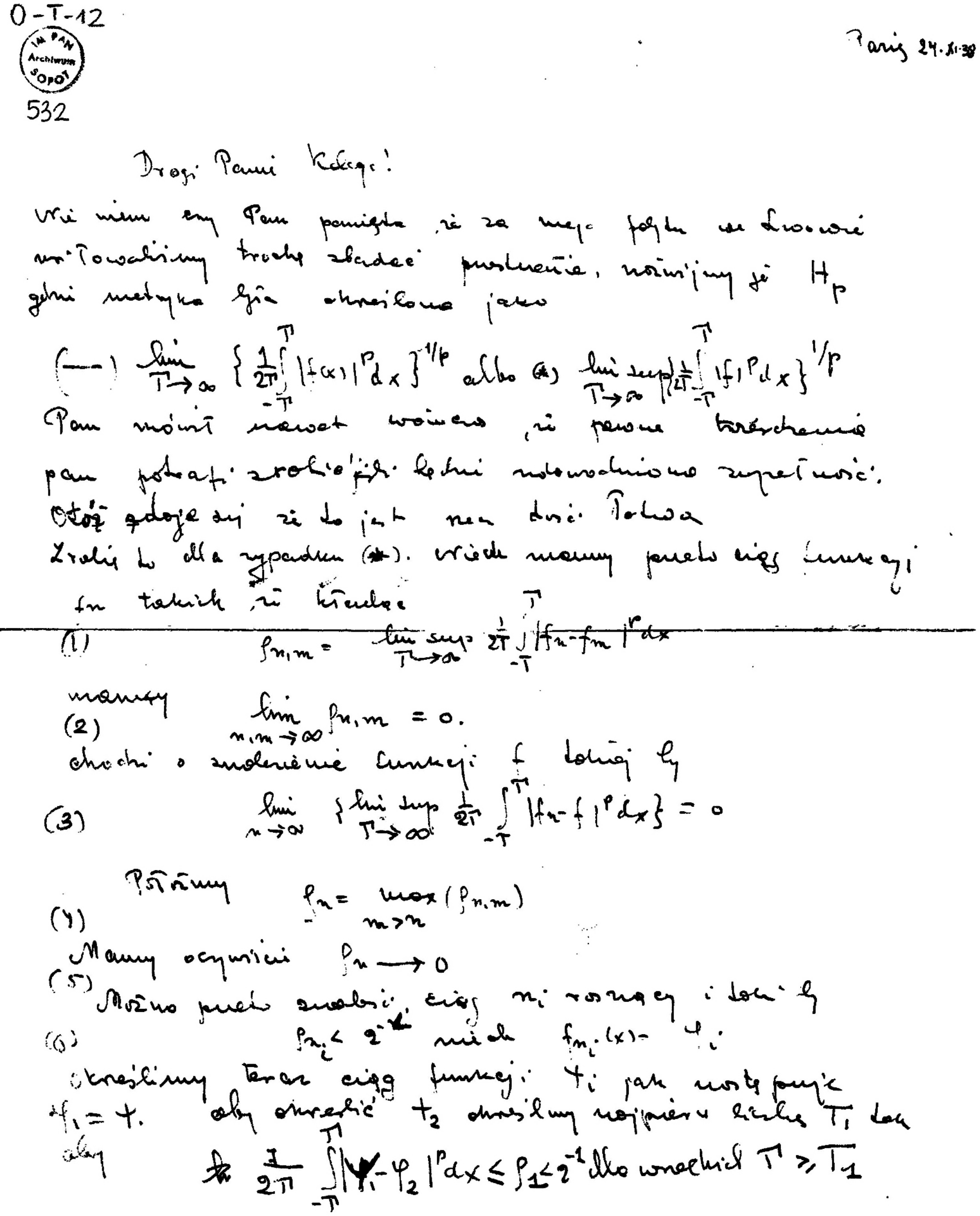
$$\|f\|_{M^p} = \limsup_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(t)|^p \, dt \right)^{1/p} < \infty. \tag{11}$$

The closure in the semi-norm $\|\cdot\|_{M^p}$ of the set of trigonometric polynomials $\sum a_k e^{i\lambda_k t}$ is the Besicovitch space of almost periodic functions (B^p a.p.) Marcinkiewicz himself was calling this space the Besicowitch space and he proved the completeness of this space in the paper [M39b] from 1939.

THEOREM 2 (Marcinkiewicz 1939). *Space $M^p(\mathbb{R})$ with semi-norm* (11) *is complete.*

Marcinkiewicz and Orlicz were trying to prove completeness in Lwów, when Marcinkiewicz was there in the academic year 1935/1936, but they were not able to do it. Two years later, on 24 November 1938, Marcinkiewicz wrote in letter from Paris to Orlicz the proof of completeness:

> *Dear Colleague, Sir!*
>
> *I don't know if You' (sic) remember that at the time of my stay in Lwów we were trying to investigate spaces, let say H_p, where the metric is defined as*
>
> $$(-)\ \lim_{T\to\infty} \left\{ \frac{1}{2T} \int_{-T}^{T} |f(x)|^p \, dx \right\}^{1/p} \quad \text{or} \quad (*)\ \limsup_{T\to\infty} \left\{ \frac{1}{2T} \int_{-T}^{T} |f(x)|^p \, dx \right\}^{1/p}.$$
>
> *You were saying at that time, that some theorems you can prove if we will have completeness. I think that this is quite simple. I will do in the case $(*)$. $(\dots)$ These spaces however will be different than spaces L^p, since they are non-separable. Therefore results, which we can get may be different than in L^p. I would like to work with you in their investigations, but unfortunately I am doing several different small things taking time and I think you will not have a great benefit from me.*
>
> *In Paris I feel great. Local mathematicians are so-so, but there are many foreigners and it is possible to have discussions about many things. Besides it is a free time from classes and, finally, it is an interesting city in terms of general culture. Best regards, signed by Marcinkiewicz.*



Paris 24.XI.38

Drogi Panie Kolego!

Nie wiem czy Pan pamięta, że za mego pobytu we Lwowie wątpiliśmy trochę składać przestrzenie, normujemy je H_p gdzie metryka była określona jako

(*) $\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T}|f(x)|^p dx\right\}^{1/p}$ albo (*) $\limsup_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T}|f|^p dx\right\}^{1/p}$

Pan mówił nawet wówczas, że pewne twierdzenie pan potrafi zrobić, gdyby była udowodniona zupełność. Otóż zdaje mi się że to jest nie dość trudne. Zrobię to dla wypadku (*). Niech mamy pewien ciąg funkcji f_n takich, że kładąc

(1) $\rho_{n,m} = \limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_n-f_m|^p dx$

mamy

(2) $\lim_{n,m\to\infty}\rho_{n,m} = 0.$

chodzi o znalezienie funkcji f takiej by

(3) $\lim_{n\to\infty}\left\{\limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_n-f|^p dx\right\} = 0$

Połóżmy

(4) $\rho_n = \max_{m>n}(\rho_{n,m})$

Mamy oczywiście

(5) $\rho_n \to 0$

Można przeto wybrać ciąg n_i rosnący i taki by

(6) $\rho_{n_i} < 2^{-i}$ niech $f_{n_i}(x) = \varphi_i$

określimy teraz ciąg funkcji ψ_i jak następuje $\psi_1 = \varphi_1$. aby określić ψ_2 określimy najpierw liczbę T_1 tak aby

$$\frac{1}{2T}\int_{-T}^{T}|\psi_1-\varphi_2|^p dx \le \rho_1 < 2^{-1} \text{ dla wszelkich } T \geq T_1$$

Photo 15. Paris, 24 Nov. 1938. The fascimile of J. Marcinkiewicz's letter to W. Orlicz (p. 1)

Note here that completeness of the space where "limsup" is replaced by "sup", in the terms of a new convergence, was given already in 1914 by the Italian mathematician Pia Nalli (1886–1964). Marcinkiewicz gave a proof of completeness for (∗) in [M39b] (see also Levitan [Le53], pp. 249–252), and other proofs were also found by Bohr and Folner [BF45, str. 54–57], Hartman and Wintner [HW47], Luxemburg and Zaanen [LZ63], and Corduneanu [Co09].

Observe that if $1 \leq p \leq q < \infty$, then

$$L^\infty(\mathbb{R}) \overset{1}{\hookrightarrow} M^q(\mathbb{R}) \overset{1}{\hookrightarrow} M^p(\mathbb{R}),$$

that is, $L^\infty(\mathbb{R}) \subset M^q(\mathbb{R}) \subset M^p(\mathbb{R})$ and $\|f\|_{M^p} \leq \|f\|_{M^q}$ for any $f \in M^q(\mathbb{R})$ and $\|f\|_{M^q} \leq \|f\|_{L^\infty}$ for any $f \in L^\infty(\mathbb{R})$.

Sometimes the name *Marcinkiewicz space* is used on the quotient space $\mathcal{M}^p = \mathcal{M}^p(\mathbb{R})$, i.e. the space $M^p(\mathbb{R})$ modulo the kernel of $\|\cdot\|_{M^p}$ (that is, $\|\cdot\|_{M^p} = 0$). This space is then a Banach ideal space on $\mathbb{R}$.

Lau (1980) investigated geometry of the ball and geometry of the Marcinkiewicz space. He proved, among other things, the following theorem:

> *The Marcinkiewicz space $\mathcal{M}_p(\mathbb{R})$ contains an isomorphic copy of l^∞. Therefore, $\mathcal{M}_p(\mathbb{R})$ is a non-separable and non-reflexive space.*

The convolution operator was investigated by Bertrandias (1966), who proved that if μ is a measure on $\mathbb{R}$ and $f \in M^p$, then the convolution $f * \mu \in M^p$ and $\|f*\mu\|_p \leq \|f\|_p \|\mu\|$. While Lau (1981) proved that if $M_r^p = \{f \in M^p\colon \lim_{T\to\pm\infty} \frac{1}{T}\int_T^{T+a} |f(t)|^p\,dt = 0\}$, then for the measure μ the norm of the convolution operator on M_r^p is equal to the norm of the convolution on $L^p(\mathbb{R})$.

Note that there are generalizations of these spaces, the so-called *Marcinkiewicz–Orlicz spaces* generated by modulars $\rho_\varphi(f) = \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi(|f(t)|)\,dt$, where $\varphi\colon [0,\infty) \to [0,\infty)$ are convex Orlicz functions. These spaces were developed by Albrycht (1956, 1959 and 1962), Wang (1959) and Hillmann (1986). In 1995 Kucher and Plichko [KP95], [KPl95] investigated also the *Marcinkiewicz-symmetric spaces* M_E defined by semi-norms $\|f\|_{M_E} = \limsup_{T\to\infty} \|f_T\|_E$, where $f_T(t) = f(tT)$ and E is an arbitrary symmetric Banach function space on $[-\frac{1}{2}, \frac{1}{2}]$. Moreover, Vo–Khac in [BCD87] and Cohen–Losert in [CL06] defined *generalized Marcinkiewicz spaces* based upon arbitrary measure spaces and limits of averages over more general families of sets.

Harmonic analysis in Marcinkiewicz spaces was investigated in [Ur61], [Be66] and [BCD87].

Marcinkiewicz spaces $M^p(\mathbb{R})$ are considered in the following books and papers:

[Le53] B. M. Levitan, *Almost-Periodic Functions*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow 1953 [5.10.1. Marcinkiewicz theorem, pp. 249–252].

[Ba84] J. Bass, *Fonctions de Corrélation, Fonctions Pseudo-Aléatoires et Applications*, Masson, Paris 1984 [Marcinkiewicz space M^2, pp. 33–35].

[Pa85] A. A. Pankov, *Bounded and Almost Periodic Solutions of Nonlinear Operator-Differential Equations*, Naukova Dumka, Kiev 1985 [Marcinkiewicz, pp. 41, 179]; English transl. Kluwer, Dordrecht 1990 [Marcinkiewicz, pp. 13, 43, 220].

[BCD87] J.-P. Bertrandias, J. Couot, J. Dhombres, M. Mendès France, V.-K. Pham Phu Hien and Kh. Vo-Khac, *Espaces de Marcinkiewicz: Corrélations, Measures, Systèmes Dynamiques*, Masson, Paris 1987 [Marcinkiewicz spaces, pp. 2–3, 11, 13, 59, 78].

[Co09] C. Corduneanu, *Almost Periodic Oscillations and Waves*, Springer, New York 2009 [Marcinkiewicz space, pp. 41–46].

[Al56] J. Albrycht, *Some remarks on the Marcinkiewicz–Orlicz space*, Bull. Acad. Polon. Sci. Cl. III. 4(1956), 1–3.

[Al59] J. Albrycht, *Some remarks on the Marcinkiewicz–Orlicz space. II, III*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7(1959), 11–12 i 55–56.

[Al62] J. Albrycht, *The theory of Marcinkiewicz–Orlicz spaces*, Dissertationes Math. Rozprawy Mat. 27(1962), 1–56.

[AG05] F. Andreano and R. Grande, *The Hilbert transform on Marcinkiewicz spaces*, African J. of Math. Physics 2(2005), No. 2, 111–118.

[ABG06] J. Andres, A. M. Bersani and R. F. Grande, *Hierarchy of almost-periodic function spaces*, Rend. Mat. Appl. (7) 26(2006), no. 2, 121–188 [Marcinkiewicz spaces, pp. 153–154].

[Ba63] J. Bass, *Espaces de Besicovitch, fonctions presque-périodiques, fonctions pseudo-aléatoires*, Bull. Soc. Math. France 91(1963), 39–61 [Marcinkiewicz theorem on completeness of M^2, p. 40].

[Be65] V. E. Beneš, *A nonlinear integral equation in the Marcinkiewicz space* $\mathfrak{m}_2$, J. Math. and Phys. 44(1965), 24–35.

[Be66] J.-P. Bertrandias, *Espaces de fonctions bornées et continues en moyenne asymptotique d'ordre p*, Bull. Soc. Math. France Mém. 5(1966), 1–106 [Besicovitch–Marcinkiewicz space M^p, pp. 12–51; Marcinkiewicz theorem on completeness, p. 12].

[BF45] H. Bohr and E. Folner, *On some types of functional spaces. A contribution to the theory of almost periodic functions*, Acta Math. 76(1945), 31–155.

[CL06] G. Cohen and V. Losert, *On Hartman almost periodic functions*, Studia Math. 173(2006), no. 1, 81–101 [Generalized Marcinkiewicz spaces, pp. 3–5].

[HW47] P. Hartman and A. Wintner, *The* (L_2)*-space of relative measure*, Proc. Nat. Acad. Sci. U. S. A. 33(1947), 128–132.

[He99] C. Heil, *The Wiener transform on the Besicovitch spaces*, Proc. Amer. Math. Soc. 127(1999), no. 7, 2065–2071.

[Hi86] T. R. Hillmann, *Besicovitch–Orlicz spaces of almost periodic functions*, in: Real and Stochastic Analysis, Wiley, New York 1986, 119–167.

[KP95] O. V. Kucher and A. M. Plichko, *Limits on the real line of symmetric spaces on segments*, Ukrain. Mat. Zh. 47(1995), no. 1, 46–55; English transl. in Ukrainian Math. J. 47(1995), no. 1, 50–62 (1996).

[KPl95] O. V. Kucher and A. M. Plichko, *The Wiener transformation on the limits of symmetric spaces*, Acta Univ. Carolin. Math. Phys. 36(1995), no. 2, 39–52.

[La80] K.-S. Lau, *On the Banach spaces of functions with bounded upper means*, Pacific J. Math. 91(1980), no. 1, 153–172.

[La81] K.-S. Lau, *The class of convolution operators on the Marcinkiewicz spaces*, Ann. Inst. Fourier (Grenoble) 31(1981), no. 3, 225–243.

[La83] K.-S. Lau, *Extension of Wiener's Tauberian identity and multipliers on the Marcinkiewicz space*, Trans. Amer. Math. Soc. 277(1983), no. 2, 489–506.

[LZ63] W. A. J. Luxemburg and A. C. Zaanen, *Notes on Banach function spaces. I*, Indag. Math. 25(1963), 135–147.

[MV08] G. Muraz and J.-L. Verger-Gaugry, *On densest packings of equal balls of* $\mathbb{R}^n$ *and Marcinkiewicz spaces*, 9 Dec. 2008, manuscript avaible at `http://arxiv.org/abs/0812.1720`

[Na14] P. Nalli, *Sopra una nuova specie di convergenza in media*, Rend. Circ. Mat. Palermo (1) 38(1914), 305–319 and 320–323.

[Ne82] R. R. Nelson, *Pointwise evaluation of Bochner integrals in Marcinkiewicz spaces*, Indag. Math. 44(1982), no. 4, 365–379.

[Sa71] U. Santarelli, *Un modello astratto dello spazio di Besicovič–Marcinkiewicz*, Boll. Un. Mat. Ital. (4) 4(1971), 907–911.

[Ur61] K. Urbanik, *Fourier analysis in Marcinkiewicz spaces*, Studia Math. 21(1961), 93–102.

[VT77] M. Vidyasagar, and M. A. L. Thathachar, *A note on feedback stability and instability in the Marcinkiewicz space* M_2, IEEE Trans. Circuits and Systems CAS-24(1977), no. 3, 127–131.

[Wa59] S. Wang, *A note on the Marcinkiewicz–Orlicz space*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7(1959), 707–710.

4.1.4. *Marcinkiewicz–Zygmund vector-valued inequalities (1939).* Marcinkiewicz–Zygmund vector-valued inequalities are estimates of operators between vector-valued spaces.

THEOREM 3 (Marcinkiewicz–Zygmund 1939). *For an arbitrary linear bounded operator $T\colon L^p \to L^p$ between real or complex (quasi-) normed Lebesgue spaces we have vector-valued estimate with constant 1, that is,*

$$\|(\sum_{k=1}^{n} |Tf_k|^2)^{1/2}\|_p \leq \|T\|_{L^p \to L^p} \|(\sum_{k=1}^{n} |f_k|^2)^{1/2}\|_p \tag{12}$$

and for $0 < p < r \leq 2$

$$\|(\sum_{k=1}^{n} |Tf_k|^r)^{1/r}\|_p \leq \|T\|_{L^p \to L^p} \|(\sum_{k=1}^{n} |f_k|^r)^{1/r}\|_p \tag{13}$$

for arbitrary $f_1, f_2, \ldots, f_n \in L^p(\mu)$ and any $n \in \mathbb{N}$.

The estimates (12) and (13) were proved by Marcinkiewicz–Zygmund with the help of Gaussian variables and r-stable Gaussian variables in paper [MZ39, Thm 1 and Thm 2]. This theorem together with their proof appeared in books by Edwards–Gaudry [EG77, pp. 203–204], Grafakos [Gr08, pp. 316–318] and in a paper by Andersen [An80]. The inequality (12) with the proof is also in the book by Nikolski [Ni02] and without proof in the book by Beckenbach and Bellman [BB83, p. 39].

More general, for an arbitrary linear bounded operator $T\colon L^p \to L^p$ between real or complex (quasi-) normed Lebesgue spaces with arbitrary σ-finite measures μ and ν, and for $0 < p, q, r \leq \infty$ and natural $n \geq 2$ let $K_{p,q}^{(n)}(r)$ be the smallest constant $C \geq 1$ in the inequality

$$\|(\sum_{k=1}^{n} |Tf_k|^r)^{1/r}\|_q \leq C \|T\|_{L^p \to L^q} \|(\sum_{k=1}^{n} |f_k|^r)^{1/r}\|_p \tag{14}$$

for arbitrary $f_1, f_2, \ldots, f_n \in L^p(\mu)$ and for fixed or any $n \in \mathbb{N}$.

The properties of constants $K_{p,q}^{(n)}(r)$ and $K_{p,q}(r) = \sup_{n \geq 2} K_{p,q}^{(n)}(r)$ for $1 \leq p, q, r \leq \infty$ were investigated by Marcinkiewicz and Zygmund (1939), Herz (1971), Krivine (1978, 1979), Defant and Floret (1993), Gasch and Maligranda (1994), Vogt (1995), Defant and Junge (1997), Maligranda and Sabourova (2011).

Equalities $K_{p,p}(2) = 1$ and $K_{p,p}(r) = 1$, where $0 < p < r \leq 2$ are just Marcinkiewicz–Zygmund results. Note that $K_{p,q}^{(n)}(r)$ are increasing in n and p, but decreasing in q. Moreover, if $0 < p \leq q \leq \infty$, then $K_{p,q}(2) = 1$.

Using the equivalence (10) we can easily prove that if $0 < p, q < \infty$ and $T\colon L^p \to L^{q,\infty}$

is a bounded linear operator, then

$$\|(\sum_{k=1}^{n} |Tf_k|^2)^{1/2}\|_{q,\infty} \leq C\|T\|_{L^p \to L^{q,\infty}} \|(\sum_{k=1}^{n} |f_k|^2)^{1/2}\|_p$$

for arbitrary $f_1, f_2, \ldots, f_n \in L^p(\mu)$ and any $n \in \mathbb{N}$. Moreover, $C \leq K_{p,r}^{(n)}(2)(\frac{q}{q-r})^{1/r}$ for any $0 < r < q$.

Constants $K_{p,q}(2)$ are connected with the relation between norm of operator $T\colon L^p_{\mathbb{R}} \to L^q_{\mathbb{R}}$ and a norm of its natural complexification between complex spaces $T_{\mathbb{C}}\colon L^p_{\mathbb{C}} \to L^q_{\mathbb{C}}$ given by formula $T_{\mathbb{C}}(f + ig) = T(f) + i\,T(g)$ (cf. [Ve76], [Kr77], [DF93], [Vo95] and [MS11]).

Among books on this topic it should be mention the following ones:

[BB83) E. F. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin 1983 [Theorem 12 of Zygmund and Marcinkiewicz, p. 30].

[DF93] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, Amsterdam 1993 [Marcinkiewicz–Zygmund result, pp. 314, 315, 347].

[EG77] R. E. Edwards and G. I. Gaudry, *Littlewood–Paley and Multiplier Theory*, Springer, Berlin-Heidelberg-New York 1977 [Marcinkiewicz–Zygmund theorem, pp. 203–204].

[GR85] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam 1985 [V.2. A theorem of Marcinkiewicz and Zygmund, pp. 482–487].

[Ni02] N. K. Nikolski, *Operators, Functions, and Systems: an Easy Reading, Vol. 1. Hardy, Hankel, and Toeplitz*, AMS, Providence 2002 [(k) Marcinkiewicz and Zygmund (1939), p. 120].

[Gr08] L. Grafakos, *Classical Fourier Analysis*, 2nd ed., Springer, New York 2008 [4.5.1. Marcinkiewicz–Zygmund theorem, pp. 316–319].

Papers discussing vector-valued Marcinkiewicz–Zygmund inequalities are:

[An80] K. F. Andersen, *Inequalities for scalar-valued linear operators that extend to their vector-valued analogues*, J. Math. Anal. Appl. 77(1980), No. 1, 264–269.

[CS83] M. Cotlar and C. Sadosky, *Vector valued inequalities of Marcinkiewicz–Zygmund and Grothendieck type for Toeplitz forms*, Lecture Notes in Math. 992(1983), 276–308.

[DJ97] A. Defant and M. Junge, *Best constants and asymptotics of Marcinkiewicz–Zygmund inequalities*, Studia Math. 125(1997), no. 3, 271–287.

[GM94] J. Gasch and L. Maligranda, *On vector-valued inequalities of the Marcinkiewicz–Zygmund, Herz and Krivine type*, Math. Nachr. 167(1994), 95–129.

[He71] C. Herz, *The theory of p-spaces with an application to convolution operators*, Trans. Amer. Math. Soc. 154(1971), 69–82.

[Kr77] J. L. Krivine, *Sur la complexification des opérateurs de L^∞ dans L^1*, C. R. Acad. Sci. Paris 284(1977), 377–379.

[Kr78] J. L. Krivine, *Constantes de Grothendieck et fonctions de type positif sur les sphères*, Séminaire sur la Géométrie des Espaces de Banach (1977–1978), École Polytech., Palaiseau 1978, Exp. No. 1–2, 1–17.

[Kr79] J. L. Krivine, *Constantes de Grothendieck et fonctions de type positif sur les sphères*, Adv. Mat. 31(1979), 16–30.

[MS11] L. Maligranda and N. Sabourova, *Real and complex operator norms between quasi-Banach $L^p - L^q$ spaces*, Math. Ineq. Appl. 14(2011), no. 2, 247–270.

[Pi11] G. Pisier, *Grothendieck's Theorem, past and present*, 26 Jan. 2011, 83 pages at `http://arxiv.org/abs/1101.4195` [Marcinkiewicz–Zygmund inequality, pp. 8–9].

[RS91] H. P. Rosenthal and S. J. Szarek, *On tensor products of operators from L^p to L^q*, Functional Analysis (Austin, TX, 1987/1989), Lecture Notes in Math. 1470, Springer, Berlin 1991, 108–132.

[Ve76] I. E. Verbickiĭ, *Some relations between the norm of an operator and that of its complex extension*, Mat. Issled. 42(1976), 3–12 (Russian).

[Vo95] H. Vogt, *Kompexifizierung von Operatoren zwischen L_p-Räumen*, Diplomarbeit, Carl von Ossietzky Universität Oldenburg, Oldenburg 1995, 1–61.

4.1.5. *Rearrangements of series – Marcinkiewicz example* (*1936*). Let X be a Banach space and $\sum_{k=1}^{\infty} x_k$ be a series in X. Denote by $S = S(\sum_{k=1}^{\infty} x_k)$ the *set of sums* of this series, that is, the set $\{x \in X\colon$ there exists a permutation $\pi\colon \mathbb{N} \to \mathbb{N}$ such that $x = \sum_{k=1}^{\infty} x_{\pi(k)}\}$.

If $X = \mathbb{R}$, then S is either empty (divergent series) or single point (absolutely convergent series) or whole $\mathbb{R}$ (for any conditionally convergent series by the Riemann theorem). If $X = \mathbb{C}$, then we have four alternatives on $S(\sum_{k=1}^{\infty} x_k)$: empty set, single point (absolutely convergent series), straight line in $\mathbb{C}$ (for example, $S = \mathbb{R} + ia$ for $\sum_{k=1}^{\infty}[\frac{(-1)^{k+1}}{k} + \frac{ia}{k(k+1)}], a \in \mathbb{R}$ fixed) and the whole $\mathbb{C}$ (if $x_{2k} \in \mathbb{R}, x_{2k-1} \in \mathbb{R}$ for any k and each of series $\sum x_{2k}, \sum x_{2k-1}$ is a conditionally convergent series).

For finite-dimensional X the famous Lévy–Steinitz theorem on rearrangements of series gives that $S(\sum_{k=1}^{\infty} x_k)$ is a linear manifold in X, that is, $S = x_0 + M$, where $x_0 \in X$ and M is a linear subspace of X. The theorem was first proven by P. Lévy in 1905. In 1913 E. Steinitz pointed out that Lévy proof was incomplete, especially in the higher-dimensional cases. Steinitz filled the gap of Lévy's proof and also found an entirely different approach (cf. [Ro87]).

Already in 1927, in the paper written by Orlicz [Or27] on page 124, it appeared Banach question on convexity of the set of sums in infinite-dimensional spaces. In 1935, in Problem 106 of the "Scottish Book", Banach asked whether Lévy–Steinitz theorem is valid in infinite-dimensional normed spaces. Banach proposed to prove that for any series in a Banach space its set of sums is a linear manifold. A simple and elegant counter-example in $L^2[0,1]$ to this conjecture was given by J. Marcinkiewicz. The solution by Marcinkiewicz also appears in the Scottish Book and the answer was negative. Marcinkiewicz constructed an example of a conditionally convergent series in infinite-dimensional Hilbert space $L^2[0,1]$ with even a nonconvex set of sums since series of integer-valued functions cannot converge in the strong L^2 topology to 1/2.

Theorem 4 (Marcinkiewicz 1936). *The Lévy–Steinitz theorem does not hold in $L^2[0,1]$ space since there exists a series in $L^2[0,1]$ such that the set of sums S is a nonconvex set.*

As the proof we present Marcinkiewicz construction ([Ma81], p. 188): in $L^2[0,1]$ consider a sequence

$$x_{2^n+k} = \chi_{[\frac{k}{2^n}, \frac{k+1}{2^n}]}, \text{where} 0 \le n < \infty, 0 \le k < 2^n.$$

Then $x_1 = \chi_{[0,1]} = 1, x_2 = \chi_{[0,\frac{1}{2}]}, x_3 = \chi_{[\frac{1}{2},1]}, x_4 = \chi_{[0,\frac{1}{4}]}, x_5 = \chi_{[\frac{1}{4},\frac{1}{2}]}, x_6 = \chi_{[\frac{1}{2},\frac{3}{4}]}, x_7 = \chi_{[\frac{3}{4},1]}, x_8 = \chi_{[0,\frac{1}{8}]}$, etc. Consider the series $\sum_{n=1}^{\infty} y_n$, where $y_{2n-1} = x_n$ and $y_{2n} = -x_n (n \ge 1)$. Since $\|x_{2^n+k}\|_2^2 = 2^{-n} \to 0$ as $n \to \infty$ it follows that $\sum_{n=1}^{\infty} y_n = (x_1 - x_1) + (x_2 - x_2) + \ldots = 0$. Also since $x_2 + x_3 - x_1 = x_4 + x_5 - x_2 = x_6 + x_7 - x_3 = \ldots = 0$ it follows that $x_1+(x_2+x_3-x_1)+(x_4+x_5-x_2)+\ldots=1$. However, no rearrangement will make that the series converge to the function identically equal to $\frac{1}{2}$ on $[0,1]$, because any of the partial sums of the series is an integer-valued function. Thus, the set of sums S is not a convex set since $0,1 \in S$ but $\frac{1}{2} \notin S$. Of course, any constant function $l, 0<l<1$ is not in S. □

Note that Marcinkiewicz construction will show nonconvexity of the set of sums S also in $L^p[0,1]$ for $0<p<\infty$, in $C[0,1]$ which was mentioned by Marcinkiewicz (since by Banach theorem [Ba32], p. 185 space $L^2[0,1]$ can be imbedded isometrically in the Banach space $C[0,1]$) and in $L^\infty[0,1]$ (since it has $C[0,1]$ as a subspace).

Examples of series in $L^p[0,1]$ with nonconvex set of sums S were given independently in 1971 by Nikishin [Ni71] (for $p=\infty$) and in 1980 by Kornilov [Ko80] (for $1 \le p \le 2$).

Kadets [Ka86], making use of the Marcinkiewicz–Kornilov example together with the Dvoretzky theorem, has shown that in any infinite-dimensional Banach space there exists a series with nonlinear set of sums S (more precisely, that the set of sums of convergent rearrangements of the series fails to be convex). Today we know more: for any fixed elements x, y of an infinite-dimensional Banach space there exists a conditionally convergent series $\sum_{k=1}^\infty x_k$ such that $S(\sum_{k=1}^\infty x_k)=\{x,y\}$ (cf. [KW89]). Moreover, for a given finite subset A of an infinite-dimensional Banach space X there is a series in X whose sum range equals A (cf. [Wo05]).

Among books, papers and bibliographies discussing Lévy–Steinits theorem, Banach problem and Marcinkiewicz example are:

[Ma81] R. D. Mauldin, *The Scottish Book. Mathematics from the Scottish Café*, Birkhäuser, Boston 1981 [Problem 106 of Banach, Marcinkiewicz example and commentary by R. D. Mauldin and W. A. Beyer, pp. 188–190].

[KK91] V. M. Kadets and M. I. Kadets, *Rearrangements of Series in Banach Spaces*, AMS, Providence 1991 [Marcinkiewicz example, pp. 24–25].

[DJT95] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Univ. Press, Cambridge 1995 [Marcinkiewicz example, p. 21].

[KK97] M. I. Kadets and V. M. Kadets, *Series in Banach Spaces. Conditional and Unconditional Convergence*, Birkhäuser, Basel 1997 [Marcinkiewicz's construction, pp. 30–31].

[Ch05] A. G. M. Champi, *Absolute and unconditional convergence in Banach spaces*, Master Thesis, Rio de Janeiro, December 2005, 1–58 (Portugese) [4.4. Marcinkiewicz counter-example, pp. 54–56].

[CC97] M.-J. Chasco and S. Chobanyan, *On rearrangements of series in locally convex spaces*, Michigan Math. J. 44(1997), no. 3, 607–617 [information on Marcinkiewicz example, p. 608].

[CG89] S. Chobanyan and G. J. Georgobiani, *A problem on rearrangements of summands in normed spaces and Rademacher sums*, Lecture Notes in Math. 1391(1989), 33–46 [information on Marcinkiewicz counter-example, p. 42].

[Ha86] I. Halperin, *Sums of a series, permitting rearrangements*, C. R. Math. Rep. Acad. Sci. Canada 8(1986), no. 2, 87–102 [9. The counter-example of Marcinkiewicz, p. 100].

[HA89] I. Halperin and T. Ando, *Bibliography: series of vectors and Riemann sums*, Hokkaido University, Research Institute of Applied Electricity, Division of Applied Mathematics, Sapporo 1989, 46 pp. [information on counter-example of J. Marcinkiewicz from 1937 in the preface].

[Ka86] V. M. Kadets, *A problem of S. Banach* (*problem 106 from the "Scottish Book"*), Funktsional. Anal. i Prilozhen. 20(1986), no. 4, 74–75; English transl. in Functional Anal. Appl.

20(1986), no. 4, 317–319.

[Ka89] V. M. Kadets, *Series permutation in infinite-dimensional spaces* (*main results and open problems*), C. R. Math. Rep. Acad. Sci. Canada 11(1989), no. 5, 151–164 [information on counter-example in $L_2[0,1]$ by J. Marcinkiewicz, p. 155].

[KW89] M. I. Kadets and K. Woźniakowski, *On series whose permutations have only two sums*, Bull. Pol. Acad. Sci. Math. 37(1989), 15–21 [information that Marcinkiewicz has given a simple counter-example, p. 15].

[Ko80] P. A. Kornilov, *On rearrangements of conditionally convergent function series*, Mat. Sb. (N. S.), 137(1980), no. 4, 598–616 (Russian).

[N71] E. M. Nikishin, *A certain problem of Banach*, Dokl. Akad. Nauk SSSR 196(1971), 774–775; English transl. in Soviet Math. Dokl. 12(1971), 255–257.

[Ni71] E. M. Nikishin, *Rearrangements of function series*, Mat. Sb. (N.S.) 85(1971), 272–285; English transl. in Math. USSR-Sb. 14(1971), 267–280.

[Or27] W. Orlicz, *Über die unabhängig von der Anordnung fast überall konvergenten Funktionenreihen*, Bull. Int. Acad. Polon. Sci. Sér. A, 1927, 117–125; Reprinted in: *Władysław Orlicz, Collected Papers I*, 41–49 [Banach problem on convexity of S on page 124].

[Ro87] P. Rosenthal, *The remarkable theorem of Lévy and Steinitz*, Amer. Math. Monthly 94(1987), no. 4, 342–351 [information on Marcinkiewicz counter-example, p. 350].

[So08] M. A. Sofi, *Levy–Steinitz theorem in infinite dimension*, New Zealand J. Math. 38 (2008), 63–73 [information on Marcinkiewicz example in $L_2[0,1]$, p. 63].

[Wo05] J. O. Wojtaszczyk, *A series whose sum range is an arbitrary finite set*, Studia Math. 171(2005), no. 3, 261–281 [information on Marcinkiewicz example in $L_2[0,1]$, p. 261].

4.2. Probability theory. In the years 1937–1938 Marcinkiewicz was interested in independent random variables. He called them independent functions. Papers, joint with Zygmund [MZ37c], [MZ38a], and his own papers [M38b], [M38c] and [M38e] are discussing problems about these functions.

4.2.1. *Marcinkiewicz–Zygmund inequalities for independent random variables* (*1937*). Consider the Rademacher functions $r_k(t) = \text{sign}[\sin(2^k\pi t)], k \in \mathbb{N}, t \in [0,1]$, which form orthonormal system in $L^2[0,1]$, that is, we have

$$\int_0^1 r_k(t)\,dt = 0, \int_0^1 r_k(t)^2\,dt = 1 \quad \text{and} \quad \int_0^1 r_k(t)\,r_m(t)\,dt = 0 \quad \text{for} \quad k \neq m.$$

Immediately from here we get equalities

$$\begin{aligned}\int_0^1 |\sum_{k=1}^n r_k(t)\,a_k|^2\,dt &= \int_0^1 [\sum_{k=1}^n r_k(t)\,a_k]\,[\sum_{m=1}^n r_m(t)\,a_m]\,dt \\ &= \sum_{k=1}^n \sum_{m=1}^n a_k a_m \int_0^1 r_k(t)\,r_m(t)\,dt \\ &= \sum_{k=1}^n \sum_{m=1}^n a_k a_m\,\delta_{km} = \sum_{k=1}^n a_k^2,\end{aligned}$$

and for L^p space the Khintchine inequality (inequalities) from 1923 reads: *for* $p \in \mathbb{R}, p > 0$ *there exist constants* $A_p, B_p > 0$ *such that*

$$A_p \left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2} \le \left(\int_0^1 |\sum_{k=1}^{n} r_k(t)\, a_k|^p \, dt\right)^{1/p} \le B_p \left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2} \tag{15}$$

for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$ *and any* $n \in \mathbb{N}$.

Rademacher functions are also independent random variables on $[0,1]$. More general, consider random variables, that is, measurable functions $X_k \colon \Omega \to \mathbb{R}$ on the probability space (Ω, Σ, P) and assume that they are independent, i.e., for any intervals $I_1, I_2, \ldots, I_n \subset \mathbb{R}$ we have the equality

$$P(\{t \in \Omega : X_1(t) \in I_1, X_2(t) \in I_2, \ldots, X_n(T) \in I_n\}) = \prod_{k=1}^{n} P(\{t \in \Omega : X_k(t) \in I_k\}).$$

Khintchine inequalities were generalized by Marcinkiewicz and Zygmund in 1937 in their paper [MZ37e] (see also [KS84], Theorems 2.5 and 2.6, and [AS10], Theorems 2 and 3) to uniformly bounded random variables on $[0,1]$.

Theorem 5 (Marcinkiewicz–Zygmund inequalities 1937). *Let* $(X_k)_{k=1}^{\infty}$ *be a sequence of independent random variables on* $[0,1]$ *satisfying conditions*

$$\int_0^1 X_k(t)\, dt = 0, \|X_k\|_2 = \left(\int_0^1 X_k(t)^2 \, dt\right)^{1/2} = 1, \|X_k\|_\infty \le M \ (k \in \mathbb{N}). \tag{16}$$

Then, for any $a = (a_k)_{k=1}^{n} \in \mathbb{R}$, *any* $n \in \mathbb{N}$ *and* $1 \le p < \infty$, *we have:*

$$m\{t \in [0,1] : |\sum_{k=1}^{n} a_k\, X_k(t)| > \lambda\, \|a\|_2\} \le 2 \exp(-\lambda^2/(4M^2)) \quad \textit{for any } \lambda > 0, \tag{17}$$

$$\frac{1}{C}(\sum_{k=1}^{n} a_k^2)^{1/2} \le \|\sum_{k=1}^{n} a_k\, X_k\|_p \le C(\sum_{k=1}^{n} a_k^2)^{1/2} \quad \textit{for some } C = C_{M,p} > 0, \tag{18}$$

and

$$\int_0^1 \exp(\lambda \max_{1 \le m \le n} |\sum_{k=1}^{m} a_k\, X_k(t)|)\, dt \le 32 \exp(\frac{1}{2}\lambda^2 \sum_{k=1}^{n} a_k^2) \quad \textit{for any } \lambda > 0. \tag{19}$$

Proof of the inequalities (17) and (18) we can find in the book [KS84, pp. 39–41]. Inequality (17) is also called the *Hoeffding inequality* (especially by probabilists and statisticians) however it was published only in 1963 by Wassily Hoeffding in [Ho63]. Note that already in 1929 estimates similar to (17) were proved by Kolmogorov [Ko29, p. 127]. A proof of estimate (19) we can find in [MZ37e], [Ka72, pp. 571–573] and [Ts51, p. 143 for Rademacher functions].

In 2000 Astashkin [As00] proved that the system of independent random variables satisfying (16) is even equivalent in the distribution sense to the Rademacher system (see also [As09], Theorem 8.4 and Corollary 8.3).

Another generalization of the Khintchine inequality for the sum of random variables was given by Marcinkiewicz and Zygmund for $p > 1$ in the paper [MZ37c, Thm 13, p. 87] from 1937 and for $p \ge 1$ in the paper [MZ38a, Thm 5, p. 109] from 1938. Falsity of the inequalities for $0 < p < 1$ was also shown by Marcinkiewicz and Zygmund in [MZ38a, pp. 112–113].

THEOREM 6 (Marcinkiewicz–Zygmund inequalities 1937). *Let $1 \le p < \infty$. Let $(X_k)_{k=1}^{\infty}$ be a sequence of independent random variables with $E(X_k) = \int_\Omega X_k(t)\, dP = 0$ and such that $E(|X_k|^p) = \int_\Omega |X_k(t)|^p\, dP < \infty$ for any $k \in \mathbb{N}$. Then there are constants $A_p^*, B_p^* > 0$ such that for any $n \in \mathbb{N}$ we have inequalities*

$$A_p^* \, \|(\sum_{k=1}^n X_k^2)^{1/2}\|_p \le \|\sum_{k=1}^n X_k\|_p \le B_p^* \, \|(\sum_{k=1}^n X_k^2)^{1/2}\|_p. \tag{20}$$

For $0 < p < 1$ no one of these two inequalities (20) is true.

Burkholder (1988) proved that for $1 < p < \infty$ we have $B_p^* \le \max(p-1, \frac{1}{p-1})$ and the constant $B_p^* = p - 1$ is sharp for $p \ge 2$.

¿From the second inequality in (20) and from the Hölder–Rogers inequality we are getting that for $p \ge 2$ we have

$$\|\sum_{k=1}^n X_k\|_p \le C_p \, n^{1/2-1/p} \, (\sum_{k=1}^n \|X_k\|_p^p)^{1/p} \tag{21}$$

or, equivalently, for $p \ge 1$

$$E(|\sum_{k=1}^n X_k|^{2p}) \le C_{2p}^{2p} \, n^{p-1} \sum_{k=1}^n E(|X_k|^{2p}). \tag{21$'$}$$

Inequality (21) or (21′) is sometimes also called *Marcinkiewicz–Zygmund inequality*, about which it was written by Petrov [Pe95, Thm 2.10, p. 62] (see also [CMR05], p. 292), and Ren–Liang [RL01, pp. 228], where they proved that $C_p \le 3\sqrt{2p}$. Combining the result of Burkholder and Ren–Liang we receive that $C_p \le \min(B_p^*, 3\sqrt{2p}\,)$.

Burkholder and Gundy (1970) generalized Marcinkiewicz–Zygmund inequality for the modular inequalities. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a convex function with $\Phi(0) = 0$, satisfying the condition Δ_2, that is, $\Phi(2u) \le C\Phi(u)$ for any $u > 0$. If $(X_k)_{k=1}^{\infty}$ is a sequence of independent random variables such that $E(X_k) = \int_\Omega X_k(t)\, dP = 0$, then there are constants $A, B > 0$ dependent only on Φ such that for any $n \in \mathbb{N}$ the following inequalities hold:

$$A \int_\Omega \Phi[(\sum_{k=1}^n X_k^2)^{1/2}]\, dP \le \int_\Omega \Phi(|\sum_{k=1}^n X_k|)\, dP \le B \int_\Omega \Phi[(\sum_{k=1}^n X_k^2)^{1/2}]\, dP. \tag{20$'$}$$

Ingenious proofs of generalized Khintchine inequality (15) for integral modular can be found in [Ka77, Lemat 6.2] and [Ma89, Sublemma 14.6(b)]: there are constants $C, D > 0$ dependent on Φ such that the following inequalities are true:

$$C\, \Phi[(\sum_{k=1}^n a_k^2)^{1/2}] \le \int_0^1 \Phi(|\sum_{k=1}^n a_k\, r_k(t)|)\, dt \le D\, \Phi[(\sum_{k=1}^n a_k^2)^{1/2}] \tag{15$'$}$$

for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and arbitrary $n \in \mathbb{N}$. Of course, the inequalities (15′) are special case of inequalities (20′).

Johnson and Schechtman (1988) proved a generalization of Marcinkiewicz–Zygmund inequalities (20) on symmetric spaces. If X is a symmetric space on $[0, 1]$, which is either separable or has the Fatou property and the lower Boyd index $\alpha_X > 0$, then the inequality of the Marcinkiewicz–Zygmund type

$$\|\sum_{k=1}^{n} X_k\|_X \le C\,\|(\sum_{k=1}^{n} X_k^2)^{1/2}\|_X \quad (n = 1, 2, \dots) \tag{22}$$

holds for any sequence of independent random variables $(X_k)_{k=1}^{\infty} \subset X$ with $\int_0^1 X_k(t)\,dt = 0 (k = 1, 2, \dots)$. In fact, Johnson and Schechtman (1988) proved that inequality (22) holds for any sequence of martingale differences $(X_k)_{k=1}^{\infty} \subset X$ if and only if the lower Boyd index $\alpha_X > 0$.

Astashkin (2008) showed that instead of assumption $\alpha_X > 0$ the necessary and sufficient condition for inequality (22) to hold is the Kruglov property of the space X, introduced by M. Sh. Braverman by using some probabilistic construction of V. M. Kruglov (1970). This property is satisfied by spaces which are sufficiently "far" from L^∞ – the smallest symmetric space on $[0, 1]$. For example, the symmetric space X has the Kruglov property if $X \supset L^p$ for some $p < \infty$ (in particular, if $\alpha_X > 0$). A more interesting example is the exponential Orlicz spaces L^M, generated by the functions $M(u) = \exp(u^p) - 1$ for $1 \le p < \infty$, which are "near" to the space L^∞ and do not contain L^q for any $q < \infty$ (Braverman 1994).

Another powerful inequality was proved by Rosenthal (1970): *Let $2 \le p < \infty$ and let $(X_k)_{k\in\mathbb{N}}$, be independent random variables such that $E(X_k) = 0$ and $E(|X_k|^p) < \infty$ for any $k \in \mathbb{N}$. Then there exist constants $C_p, D_p > 0$ such that for any $n \in \mathbb{N}$ the following inequalities hold:*

$$\begin{aligned} C_p \max\left((\sum_{k=1}^{n} \|X_k\|_{L^p}^p)^{1/p}, (\sum_{k=1}^{n} \|X_k\|_{L^2}^2)^{1/2} \right) &\le \|\sum_{k=1}^{n} X_k\|_{L^p} \\ &\le D_p \max\left((\sum_{k=1}^{n} \|X_k\|_{L^p}^p)^{1/p}, (\sum_{k=1}^{n} \|X_k\|_{L^2}^2)^{1/2} \right). \end{aligned}$$

The Marcinkiewicz–Zygmund inequalities (20) can be found, e.g., in the books by Kawata ([Ka72], Thm 13.6.1) and Gut ([Gu05], Thm 8.1). Gut, in fact, gave the proof of the Marcinkiewicz–Zygmund inequalities with the help of Khintchine inequality with constants $A_p^* = A_p^{1/p}/2, B_p^* = 2B_p^{1/p}$ (pp. 150–151), and the Rosenthal inequality is proved with the help of Marcinkiewicz–Zygmund inequalities (pp. 151–153).

Marcinkiewicz and Zygmund proved also other inequalities in the papers [MZ37c] and [MZ38a, Thm 5]. For example, in the first paper in Theorems 1 and 3 we have the following estimates:

Theorem 7 (Marcinkiewicz–Zygmund inequalities 1937). *Let X_k be independent random variables such that $E(X_k) = 0$ for $k = 1, 2, \dots, n$ and $S_n = \sum_{k=1}^{n} X_k$.*

(a) *If $p > 1$, then*

$$\|\max_{1\le m\le n} |S_m|\,\|_p \le 2^{1/p} \frac{p}{p-1}\,\|S_n\|_p. \tag{23}$$

(b) *If $EX_k^2 = 1$ and $E|X_k| \ge \alpha > 0$ for $k = 1, 2, \dots, n$, then there exists a constant*

$C = C(\alpha) > 0$ such that for any $(a_k)_{k=1}^n$ we have

$$\| \max_{1\le m\le n} |\sum_{k=1}^{m} a_k\, X_k|\|_1 \le C(\alpha)\, \| \sum_{k=1}^{n} a_k\, X_k\|_1. \tag{24}$$

In the proof of Theorem 7 (a) they are using the Hardy–Littlewood result (1930) on the maximal function. In the proof of part (b) they use the result from (a) for $p = 2$ and the Paley–Zygmund inequality ([PZ32], str. 192). Moreover, Mogyoródi [Mo79] generalized the inequality (23) to the form $\| \max_{1\le m\le n} \Phi(|S_m|)\,\|_1 \le C\, \|\Phi(|S_n|)\|_1$, where Φ is a convex Young function satisfying together with its complementary function the so-called Δ_2-condition.

Let us collect some books and papers quoted above or devoted to this subject:

[Ka72] T. Kawata, *Fourier Analysis in Probability Theory*, Academic Press, New York-London 1972 [13.5.3. Marcinkiewicz–Zygmund theorem, pp. 568–571; 13.6.1. Marcinkiewicz–Zygmund, pp. 576–578].

[St74] W. F. Stout, *Almost Sure Convergence*, Academic Press, New York-London 1974 [Theorem 3.3.6. Martingale version of Marcinkiewicz and Zygmund inequality, pp. 149–152].

[Pe75] V. V. Petrov, *Sums of Independent Random Variables*, Springer, New York 1975 [pp. 59–60].

[KS84] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow 1984 (Russian) [Marcinkiewicz–Zygmund inequalities (18), (17): Theorems 2.5 and 2.6, pp. 39–42].

[SW86] G. R. Shorack and J. A. Wellner, *Empirical Processes with Applications to Statistics*, Wiley, New York 1986 [Appendix 5. Marcinkiewicz and Zygmund equivalences, pp. 858–859].

[CT88] Y. S. Chow and H. Teicher, *Probability Theory. Independence, Interchangeability, Martingales*, 2nd ed., Springer, New York 1988 [10.3 Marcinkiewicz–Zygmund inequality, pp. 366–369; 11.2. Martingale extension of Marcinkiewicz–Zygmund inequalities, pp. 394–402].

[Pe95] V. V. Petrov, *Limit Theorems of Probability Theory. Sequences of Independent Random Variables*, Oxford Univ. Press, New York 1995 [2.6.18. Marcinkiewicz–Zygmund inequalities, pp. 82; Thm 2.10 is a corollary from Marcinkiewicz–Zygmund inequalities, pp. 62 and 77].

[DG99] V. H. De la Peña and E. Giné, *Decoupling. ¿From Dependence to Independence. Randomly Stopped Processes. U-statistics and Processes. Martingales and Beyond*, Springer, New York 1999 [Lemma 1.4.13. Marcinkiewicz inequalities, pp. 34–35].

[CMR05] O. Cappé, E. Moulines and T. Rydén, *Inference in Hidden Markov Models*, Springer, New York 2005 [9.1.5. Marcinkiewicz–Zygmund inequality, p. 292].

[Gu05] A. Gut, *Probability: A Graduate Course*, Springer Texts in Statistics, Springer, New York 2005 [8.1. Marcinkiewicz–Zygmund inequalities, pp. 150–151].

[As00] S. V. Astashkin, *Systems of random variables equivalent in distribution to the Rademacher systems, and the $\mathcal{K}$-closed representability of Banach pairs*, Mat. Sb. 191(2000), no. 6, 3–30; English transl. in Sb. Math. 191(2000), no. 5–6, 779–807.

[As08] S. V. Astashkin, *Independent functions in symmetric spaces and the Kruglov property*, Mat. Sb. 199(2008), no. 7, 3–20; English transl. in: Sb. Math. 199(2008), no. 7–8, 945–963.

[As09] S. V. Astashkin, *Rademacher functions in symmetric spaces*, Sovrem. Mat. Fundam. Napravl. 32(2009), 3–161; English transl. in J. Math. Sci., New York 169(2010), no. 6, 725–886.

[AS10] S. V. Astashkin and F. A. Sukochev, *Independent functions and geometry of Banach spaces*, Uspekhi Mat. Nauk 65(2010), no. 6, 3–86; English transl. in Russian Math. Surveys 65(2010), no. 6, 1003–108 [Marcinkiewicz–Zygmund inequalities (18), (17): Theorems 2 and 3].

[BBLM] S. Boucheron, O. Bousquet, G. Lugosi and P. Massart, *Moment inequalities for functions of independent random variables*, Ann. Probab. 33(2005), no. 2, 514–560.

[Bu88] D. L. Burkholder, *Sharp inequalities for martingales and stochastic integrals*, Asterisque 157–158(1988), 75–94.

[BG70] D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. 124(1970), 249–304.

[Eg07] V. A. Egorov, *Estimation of constants in the right-hand side of the inequalities of Marcinkiewicz and Rosenthal*, Zap. Nauchn. Sem. POMI 341(2007), 115–123; English transl. in J. Math. Sci. (N. Y.) 147(2007), no. 4, 6912–6917.

[Eg09] V. A. Egorov, *Two-sided estimates for constants in the Marcinkiewicz inequalities*, Zap. Nauchn. Sem. POMI 361(2008), 45–56; English transl. in J. Math. Sci. (N. Y.) 159(2009), no. 3, 305–311 [estimation of the constant B_p^* in (20)].

[Gu67] R. F. Gundy, *The martingale version of a theorem of Marcinkiewicz and Zygmund*, Ann. Math. Statist. 38(1967), 725–734.

[Ho63] W. Hoeffding, *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. 58(1963), 13–30.

[Ho97] P. Holgate, *Independent functions: Probability and analysis in Poland between the Wars*, Biometrika 84(1997), no. 1, 161–173.

[IS99] R. Ibragimov and Sh. Sharakhmetov, *Analogues of Khintchine, Marcinkiewicz–Zygmund and Rosenthal inequalities for symmetric statistics*, Scand. J. Statist. **26**(1999). no. 4, 621–633.

[JS88] W. B. Johnson and G. Schechtman, *Martingale inequalities in rearrangement invariant function spaces*, Israel J. Math. 64(1988), no. 3, 267–275.

[JSZ85] W. B. Johnson, G. Schechtman and J. Zinn, *Best constants in moment inequalities for linear combinations of independent and exchangeable random variables*, Ann. Probab. 13(1985), 234–253.

[Ka77] N. J. Kalton, *Orlicz sequence spaces without local convexity*, Math. Proc. Cambridge Philos. Soc. 81(1977), no. 2, 253–277.

[Kh23] A. Khintchine, *Über dyadische Brüche*, Math. Z. 18(1923), no. 1, 109–116.

[Ko29] A. Kolmogoroff, *Über das Gesetz des Logarithmus*, Math. Ann. 101(1929), 126–135.

[Kr02] A. Krantsberg, *Khinchin and Marcinkiewicz–Zygmund-type inequalities for quadratic forms of independent random variables*, Statist. Probab. Lett. 60(2002), no. 3, 321–327 [Marcinkiewicz–Zygmund inequalities (18), (17): pages 321–322].

[Ma89] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Mathematics 5, Universidade Estadual de Campinas, Campinas 1989.

[Mo79] J. Mogyoródi, *On an inequality of Marcinkiewicz and Zygmund*, Publ. Math. Debrecen 26(1979), no. 3–4, 267–274.

[Na00] S. V. Nagaev, *Probabilistic and moment inequalities for dependent random variables*, Teor. Veroyatnost. i Primenen. 45(2000), no. 1, 194–202; English transl. in Theory Probab. Appl. 45(2001), no. 1, 152–160.

[PZ32] R. E. A. C. Paley and A. Zygmund, *On some series of functions. III*, Proc. Cambridge Phil. Soc. 28(1932), 190–205.

[RL01] Y.-F. Ren and H.-Y. Liang, *On the best constant in Marcinkiewicz–Zygmund inequality*, Statist. Probab. Lett. 53(2001), no. 3, 227–233.

[Ri09] E. Rio, *Moment inequalities for sums of dependent random variables under projective conditions*, J. Theoret. Probab. 22(2009), no. 1, 146–163.

[Ro70] H. P. Rosenthal, *On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables*, Israel J. Math. 8(1970), 273–303.

[Ts51] T. Tsuchikura, *Notes on Fourier analysis. XL. Remark on the Rademacher system*, Proc. Japan Acad. 27(1951), 141–145.

4.2.2. *Marcinkiewicz–Zygmund strong law of large numbers and random series.* In the paper [MZ37c] on independent random variables Marcinkiewicz and Zygmund generalized the classical Kolmogorov strong law of large numbers (1933) on any $p \in (0, 2)$ (Kolmogorow proved it for $p = 1$): *Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with the same distribution. Denote $S_n = \sum_{k=1}^n X_k, n \geq 1$. Then $\frac{S_n - nc}{n} \to 0$ almost surely (i.e. with probability 1 or almost everywhere) for some $c \in \mathbb{R}$ if and only if $E|X_1| < \infty$, in which case $c = EX_1$.*

Theorem 8 (Marcinkiewicz–Zygmund strong law of large numbers 1937). *Let $0 < p < 2$ and let $X_1, X_2, \ldots$ be a sequence of independent random variables with the same distribution.*

(*a*) *If $E|X_1|^p < \infty$, then $\frac{S_n - nc}{n^{1/p}} \to 0$ almost surely, where $c = EX_1$ for $1 \leq p < 2$ and any $c \in \mathbb{R}$ for $0 < p < 1$.*
(*b*) *If $\frac{S_n - nc}{n^{1/p}} \to 0$ with probability 1 for some $c \in \mathbb{R}$, then $E|X_1|^p < \infty$.*

The classical Marcinkiewicz–Zygmund theorem appeared in the following monographs:

[Lo63] M. Loéve, *Probability Theory*, Third Edition, Van Nostrand, Princeton 1963 [Kolmogorov: $p = 1$; Marcinkiewicz: $p \neq 1$, pp. 242–243].

[St74] W. F. Stout, *Almost Sure Convergence*, Academic Press, New York-London 1974 [Theorem 3.2.3 is due to Marcinkiewicz, pp. 126–128].

[LT91] M. Ledoux and M. Talagrand, *Probability in Banach Spaces. Isoperimetry and Processes*, Springer-Verlag, Berlin 1991 [Marcinkiewicz–Zygmund theorem on random variables with values in Banach space B, Thm. 7.9, pp. 186–190; relation between type of a Banach space B and Marcinkiewicz–Zygmund theorem on random variables with values in B, Thm 7.9, Thm 9.21, pp. 259–260].

[Ka97] O. Kallenberg, *Foundations of Modern Probability*, Springer, New York 1997 [Theorem 3.23. Marcinkiewicz and Zygmund with proof, p. 51].

[Gu05] A. Gut, *Probability: A Graduate Course*, Springer Texts in Statistics, Springer, New York 2005 [Part 6.7. The Marcinkiewicz–Zygmund strong law and Thm 7.1. The Marcinkiewicz–Zygmund strong law with proof, pp. 298–301].

[AL06] K. B. Athreya and S. N. Lahiri, *Measure Theory and Probability Theory*, Springer Texts in Statistics. Springer, New York 2006 [Part 8.4. Kolmogorov and Marcinkiewicz–Zygmund SLLNs], Theorem 8.4.4. (Marcinkiewicz–Zygmund SLLNs) with proof].

[BW07] R. Bhattacharya and E. C. Waymire, *A Basic Course in Probability Theory*, Springer, New York 2007 [Thm 9.5 of Marcinkiewicz and Zygmund (1937) with the proof, pp. 124–126].

Theorem 8 appeared also in the following papers:

[HL92] E. Hensz and A. Łuczak, *Marcinkiewicz strong law of large numbers*, in: “Probability and Mechanics in the Historical Sketches”, Proc. of the 5th All-Polish School on the History of Mathematics (Dziwnów, 9–13 May 1991), Edited by Stanisław Fudali, Part 1. Probability, Szczecin 1992, 219–222 (Polish).

[MP10] I. K. Matsak and A. M. Plichko, *On the Marcinkiewicz–Zygmund law of large numbers in Banach lattices*, Ukrain. Mat. Zh. 62(2010), no. 4, 504–513; English transl. in Ukrainian Math.

J. 62(2010), no. 4, 575–587.

[Ne98] D. Neuenschwander, *The Marcinkiewicz–Zygmund law of large numbers on the group of Euclidean motions and the diamond group*, J. Math. Sci. (New York) 89(1998), no. 5, 1535–1540.

[Sz10] Z. S. Szewczak, *Marcinkiewicz laws with infinite moments*, Acta Math. Hungar. 127 (2010), 64–84.

[Sz11] Z. S. Szewczak, *On Marcinkiewicz–Zygmund laws*, J. Math. Anal. Appl. 375(2011), no. 2, 738–744.

[Sz92] D. Szynal, *History of strong law of large numbers until 1939*, in: "Probability and Mechanics in the Historical Sketches", Proc. of the 5th All-Polish School on the History of Mathematics (Dziwnów, 9–13 May 1991), Edited by Stanisław Fudali, Part 1. Probability, Szczecin 1992, 120–176 (Polish).

Strong law of large numbers of Marcinkiewicz–Zygmund was generalized to random variables with values in Banach spaces and is closely connected with Rademacher type of a Banach space. For $p \in [1,2]$ one says that a Banach space B has Rademacher *type p* if there is some constant $C > 0$ such that $\left(\int_0^1 \|\sum_{k=1}^n x_k\, r_k(t)\|_B^p\, dt\right)^{1/p} \le C(\sum_{k=1}^n \|x_k\|_B^p)^{1/p}$ for all $x_1, x_2, \ldots, x_n \in B$ and any $n \in \mathbb{N}$. Any Banach space has type 1; Hilbert spaces have type 2; for $p < q$, type q implies type p. Beck [Be62] found a necessary and sufficient condition (Beck convexity, or "B-convexity") on the geometry of B that for every B-valued independent random variables $X_1, X_2, \ldots$ with mean zero and $\sup_n E\|X_n\|_B^2 < \infty$ we have $\frac{X_1+X_2+\ldots+X_n}{n} \to 0$ as $n \to \infty$. The B-convexity condition holds if and only if B has type p for some $p > 1$.

The following extension of the Marcinkiewicz–Zygmund SLLN to B-valued random variables was proved by de Acosta [Ac81, p.160] (see also Azlarov and Volodin [AV81]): *Let $1 \le p < 2$. A Banach space B has the Rademacher type p if and only if for every B-valued sequence of identically distributed independent random variables $(X_k)_{k=1}^\infty$ with mean zero and $E\|X_1\|_B^p < \infty$ satisfies the SLLN: $\lim_{n\to\infty} \frac{X_1+X_2+\ldots+X_n}{n^{1/p}} = 0$ almost surely.*

The Marcinkiewicz–Zygmund law of large numbers was examined by Marcus and Woyczyński [MW79] in Banach spaces of stable type, Woyczyński [Wo81], Korzeniowski [Ko84], Bingham [Bi86] and many others. More information can be found in the papers collected below and in the mentioned monograph by Ledoux and Talagrand (1991).

[Ac81] A. de Acosta, *Inequalities for B-valued random vectors with applications to the strong law of large numbers*, Ann. Probab. 9(1981), no. 1, 157–161.

[AV81] T. A. Azlarov and N. A. Volodin, N. A. *Laws of large numbers for identically distributed Banach space valued random variables*, Teor. Veroyatnost. i Primenen. 26(1981), no. 3, 584–590; English transl. in Theory Prob. Appl. 26(1981), no. 3, 573–580.

[Be62] A. Beck, *A convexity condition in Banach spaces and the strong law of large numbers*, Proc. Amer. Math. Soc. 13(1962), 329–334.

[Bi86] N. H. Bingham, *Extensions of the strong law*, Adv. in Appl. Probab. 18(1986), suppl., 27–36 [Marcinkiewicz–Zygmund LLN, p. 29].

[GZ92] E. Giné and J. Zinn, *Marcinkiewicz type law of large numbers and convergence of moments for U-statistics*, in: "Probability in Banach Spaces 8" (Proc. Eighth Internat. Conference held at Bowdoin College, Brunswick, Maine, 1991), Edited by R. M. Dudley, M. G. Hahn and J. Kuelbs, Birkhaüser, Boston 1992, 273–291.

[HH10] F. Hechner and B. Heinkel, *The Marcinkiewicz–Zygmund LLN in Banach spaces: A generalized martingale approach*, J. Theor. Probab. 23(2010), no. 2, 509–522.

[Ko84] A. Korzeniowski, *On Marcinkiewicz SLLN in Banach spaces*, Ann. Probab. 12(1984), no. 1, 279–280.

[MW79] M. B. Marcus and W. A. Woyczyński, *Stable measures and central limit theorem in spaces of stable type*, Trans. Amer. Math. Soc. 271(1979), 71–102.

[Ri95] E. Rio, *A maximal inequality and dependent Marcinkiewicz–Zygmund strong laws*, Ann. Probab. 23(1995), no. 2, 918–937.

[Su93] Z. G. Su, *Marcinkiewicz laws of large numbers for a sequence of independent Banach space-valued random variables*, Acta Math. Sinica 36(1993), no. 6, 731–739 (Chinese).

[Su96] Z. Su, *The law of the iterated logarithm and Marcinkiewicz law of large numbers for B-valued U-statistics*, J. Theoret. Probab. 9(1996), no. 3, 679–701.

[ST92] K. L. Su and R. L. Taylor, *Marcinkiewicz strong laws of large numbers and convergence rates for arrays of independent random elements in Banach spaces*, Stochastic Anal. Appl. 10(1992), no. 2, 223–237.

[Wo80] W. A. Woyczyński, *On Marcinkiewicz–Zygmund laws of large numbers in Banach spaces and related rates of convergence*, Probab. Math. Statist. 1(1980), no. 2, 117–131.

Frequent method of proof of the strong law of large numbers is to demonstrate the convergence almost surely of some random series and to use the Kronecker lemma. Sufficient conditions or criteria of convergence were given by Khintchine and Kolmogorov (1925). If $(X_n)_{n=1}^{\infty}$ is a sequence of independent random variables, then by zero-one law of Kolmogorov the probability that the series $\sum_{n=1}^{\infty} X_n$ is convergent is equal either to 0 or 1.

Marcinkiewicz and Zygmund in the paper [MZ37c] proved also the following theorem on random series:

Theorem 9 (Marcinkiewicz–Zygmund 1937). (a) *Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent random variables such that $EX_n = 0, E(X_n^2) = 1, n \geq 1$ and $\inf_{n\in\mathbb{N}} E|X_n| > 0$. If the series $\sum_{n=1}^{\infty} a_n X_n$ is almost surely convergent for a sequence $(a_n)_{n=1}^{\infty}$ of real numbers, then $\sum_{n=1}^{\infty} a_n^2 < \infty$.*

(b) *If $(X_n)_{n=1}^{\infty}$ is a sequence of independent random variables with the same distribution such that $E|X_1|^p < \infty$ for some $0 < p < 2$, then*

$$\sum_{n=1}^{\infty} \frac{X_n - EY_n}{n^{1/p}} < \infty \quad \textit{almost surely},$$

where $Y_n = X_n \mathrm{I}_{\{|X_n| \leq n^{1/p}\}}$. Morever, if either $0 < p < 1$ or $1 < p < 2$ and $EX_1 = 0$, then the series $\sum_{n=1}^{\infty} \frac{X_n}{n^{1/p}}$ is convergent almost surely.

In 1938 Marcinkiewicz and Zygmund proved in the paper [MZ38a] the next famous result for real-valued random variables (see Theorem 10). Kahane in the book [Ka85] generalized this result to random variables with values in a Banach space X.

Assume that the infinite matrix of numbers $T = (a_{nm})_{n,m=1}^{\infty}$ is a summability matrix (Toeplitz matrix), that is, it satisfies the condition $\lim_{n\to\infty} a_{nm} = 1$ for $m = 1, 2, \ldots$. A sequence $x = (x_n), x_n \in X$ is T-summable, if the series $T_n(x) = \sum_{m=1}^{\infty} a_{nm} x_m$ are convergent in X for each $n \in \mathbb{N}$ and the sequence $\{T_n(x)\}$ is convergent in X.

THEOREM 10 (Marcinkiewicz–Zygmund 1938). *Let* $(X_n)_{n=1}^{\infty}$ *be a sequence of independent random variables with values in a Banach space* X *and let* T *be a matrix of summability. If the series* $\sum_{n=1}^{\infty} X_n$ *is almost surely* T*-summable, then there exists a sequence* $x_n \in X$ *such that the series* $\sum_{n=1}^{\infty}(X_n - x_n)$*is convergent almost surely. If the series* $\sum_{n=1}^{\infty} X_n$ *is almost surely* T*-bounded, then there exists a sequence* $x_n \in X$ *such that the series* $\sum_{n=1}^{\infty}(X_n - x_n)$ *is bounded almost surely.*

Another proof of Theorem 10 together with necessity of conditions in this theorem on convergence for the case $X = \mathbb{R}$ was given by Tucker [Tu65]. Kahane says in [Ka85] that P. Lévy has such a theorem in his paper [Le35], which I cannot see. The formulation maybe is in Léy's book [Le37].

On almost sure convergence of random series we can find in many books. For example, books listed below contain Marcinkiewicz–Zygmund theorems (Theorem 9: [CT88], [KS84], [St74] and Theorem 10: [Ka63], [Ka85]):

[Le37] P. Lévy, *Théorie de l'Addition des Variables Alèatories*, Gauthier-Villars, Paris 1937.

[Ka63] J.-P. Kahane, *Séries de Fourier aléatoires*, Séminaire de Math. Supérieures, No. 4 (1963), Université de Montréal, Montreal 1967 [Thm 5, pp. 39–41].

[Ka72] T. Kawata, *Fourier Analysis in Probability Theory*, Academic Press, New York-London 1972 [13.7.1 and 13.8.1. Marcinkiewicz–Zygmund, pp. 583–584 and 588–590].

[KS84] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow 1984 (Russian).

[Ka85] J.-P. Kahane, *Some Random Series of Functions*, 2nd ed., Cambridge Univ. Press, Cambridge 1985 [Thm 2, pp. 13–17].

[CT88] Y. S. Chow and H. Teicher, *Probability Theory. Independence, Interchangeability, Martingales*, 2nd ed., Springer, New York 1988 [Theorem 3 (Marcinkiewicz–Zygmund), p. 118].

[LW83] T. L. Lai and C. Z. Wei, *A note on martingale difference sequences satisfying the local Marcinkiewicz–Zygmund condition*, Bull. Inst. Math. Acad. Sinica 11(1983), no. 1, 1–13.

[Le35] P. Lévy, *Sur la sommabilité des séries aléatoires divergentes*, Bull. Soc. Math. France 63(1935), 1–35.

[Tu65] H. G. Tucker, *On quasi-convergence of series of independent random variables*, Proc. Amer. Math. Soc. 16(1965), 435–439.

4.2.3. *Law of the iterated logarithm (1937).* In 1929 Kolmogorov proved the so-called law of the iterated logarithm: *Let* (X_n) *be a sequence of independent random variables each with mean zero and finite variance. Let* $S_n = \sum_{k=1}^{n} X_k$ *and* $B_n = \sum_{k=1}^{n} E(X_k^2) \to \infty$*, when* $n \to \infty$*. If there exists a sequence* $(M_n)_{n=1}^{\infty}$ *of positive numbers such that*

$$|X_n| \le M_n \quad \text{and} \quad M_n = o\left((\frac{B_n}{\log\log B_n})^{1/2} \right), \tag{25}$$

i.e. $X_n = o\left((\frac{B_n}{\log\log B_n})^{1/2} \right)$ *almost surely, then*

$$\mathrm{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1 \right) = 1, \tag{26}$$

that is, $\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1$ *almost surely.*

The second assumption in condition (25) cannot be weakened, which was proved by Marcinkiewicz and Zygmund in their paper [MZ37e] from 1937.

THEOREM 11 (Marcinkiewicz–Zygmund 1937). *There exists a sequence $(X_n)_{n=1}^{\infty}$ of independent (two-valued) random variables such that $EX_k = 0, \sigma^2 X_k < \infty$ for $k \geq 1$ and*

$$M_n := \max_{1\leq k\leq n} |X_k| = O\Big((\frac{B_n}{\log\log B_n})^{1/2} \Big) \quad \textit{and} \quad \mathrm{P}\Big(\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n\log\log n}} = 1 \Big) = 0,$$

i.e., $\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} < 1$ *almost surely.*

It is worth to mention a remark of Marcinkiewicz to Zygmund, which is not in the joint paper with Zygmund (only in the overview of Marcinkiewicz results [Zy60], pp. 38): for any sequence of numbers $(a_n)_{n=1}^{\infty}$ such that $B_n = \sum_{k=1}^{n} a_k \to \infty$ as $n \to \infty$ we have

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} a_k\, r_k(t)}{\sqrt{2B_n\, \log\log B_n}} \leq 1$$

for almost all points t from the interval $[0, 1]$. The proof follows from classical argument of Kolmogorov.

Let us note that Hartman and Winter [HW41] showed in 1941 that if $(X_n)_{n=1}^{\infty}$ is a sequence of independent random variables with the same distribution such that $E(X_1) = \mu$ and $\sigma^2(X_1) = \sigma^2 < \infty$, then

$$\mathrm{P}\Big(\liminf_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{2n\log\log n}} = -1 \Big) = \mathrm{P}\Big(\limsup_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{2n\log\log n}} = 1 \Big) = 1.$$

¿From here can be derived that with probability 1 the set of all limit points of the sequence $(\frac{S_n - n\mu}{\sigma\sqrt{2n\log\log n}})_{n=3}^{\infty}$ is the interval $[-1, 1]$.

Strassen [St66] proved a converse theorem to the law of iterated logarithm: if $(X_n)_{n=1}^{\infty}$ is a sequence of independent random variables with the same distribution and

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1,$$

then $EX_1 = 0$ and $EX_1^2 = 1$.

A survey article on law of the iterated logarithm was published by Bingham [Bi86]. Moreover, these problems appeared in the following books and papers:

[St74] W. F. Stout, *Almost Sure Convergence*, Academic Press, New York-London 1974 [Part 5.2].

[Pe95] V. V. Petrov, *Limit Theorems of Probability Theory: Sequences of Independent Random Variables*, Oxford Univ. Press, New York 1995.

[Bi86] N. H. Bingham, *Variants on the law of the iterated logarithm*, Bull. London Math. Soc. 18(1986), no. 5, 433–467.

[Bi00] N. H. Bingham, *Studies in the history of probability and statistics XLVI. Measure into probability: From Lebesgue to Kolmogorov*, Biometrika 87(2000), No. 1, 145–156.

[Fe45] W. Feller, *The fundamental limit theorems in probability*, Bull. Amer. Math. Soc. 51(1945), no. 11, 800–832.

[Ga66] V. F. Gaposhkin, *Lacunary series and independent functions*, Uspekhi Mat. Nauk 21(1966), no. 6, 3–82; English transl. in Russian Math. Surveys 21(1966), 1–82.

[Ha41] P. Hartman, *Normal distributions and the law of the iterated logarithm*, Amer. J. Math. 63(1941), no. 3, 584–588

[HW41] P. Hartman and A. Wintner, *On the law of the iterated logarithm*, Amer. J. Math. 63(1941), 169–176.

[Ko29] A. Kolmogorov, *Über das Gesetz des Iterierten Logarithmus*, Math. Ann. 101(1929), 126–135.

[St66] V. Strassen, *A converse to the law of the iterated logarithm*, Z. Wahrsch. Verw. Gebiete 4(1966), 265–268.

[Sz92] D. Szynal, *History of strong law of large numbers until 1939*, in: "Probability and Mechanics in the Historical Sketches", Proceedings of the 5th All-Polish School on the History of Mathematics held in Dziwnów, 9–13 May 1991, Edited by Stanisław Fudali, Part 1. Probability, Szczecin 1992, 120–176 (Polish).

[To71] R. J. Tomkins, *Some iterated logarithm results related to the central limit theorem*, Trans. Amer. Math. Soc. 156(1971), 185–192.

After Kolmogorov law of iterated logarithm different forms of this theorem became an object of interest of several mathematicians (see Feller [Fe43] and the references given there). Some bounds for the sums $S_n = \sum_{k=1}^{n} X_k$ of a sequence of independent random variables $(X_k)_{k=1}^{\infty}$ were proved in 1931 by Lévy [Le31] for $0 < \alpha < 1$ using the stable distribution. The method does not work for $\alpha \geq 1$ and he formulated the question if his result is also true for $1 \leq \alpha < 2$. Marcinkiewicz [M39k] gave a positive answer in 1939. Afterwards this result was named the *Lévy–Marcinkiewicz theorem* (cf. Feller [Fe46], pp. 257–258): *Let $(X_k)_{k=1}^{\infty}$ be a sequence of independent random variables. Suppose that one has estimates uniformly for large u and all $k \in \mathbb{N}$*

$$c\,u^{-\alpha} \leq P(|X_k| > u) \leq C\,u^{-\alpha},$$

where c, C are positive constants and $0 < \alpha < 1$. Let $\lambda(t)$ be an increasing function such that $\lim_{t\to\infty} \frac{\lambda(2t)}{\lambda(t)} = 1$. Then the probability for an infinite number of realizations of the inequality

$$|S_n| > n \log n \lambda(\log n)^{1/\alpha}$$

is zero (one) if the series $\sum_{n=1}^{\infty} \frac{1}{n\lambda(n)}$ converges (diverges). The theorem remains true also for $1 \leq \alpha < 2$ provided that $E(X_k) = 0$.

Some generalizations of the Lévy–Marcinkiewicz result were given by Feller [Fe43], [Fe45], [Fe46], Kunisawa [Ku49b], [Ku49], Lipschutz [Li56] and later generalizations obtained by many authors received the name of Feller.

[Ku49b] K. Kunisawa, *Limit Theorems in Probability Theory*, Chubunkan, Tokyo 1949 (Japanese) [Chapter 10, generalization of the Lévy–Marcinkiewicz theorem].

[Fe43] W. Feller, *The general form of the so-called law of the iterated logarithm*, Trans. Amer. Math. Soc. 54(1943), 373–402 [Lévy–Marcinkiewicz result, p. 377].

[Fe45] W. Feller, *The fundamental limit theorems in probability*, Bull. Amer. Math. Soc. 51(1945), 800–832 [Lévy and Marcinkiewicz result, p. 809].

[Fe46] W. Feller, *A limit theorem for random variables with infinite moments*, Amer. J. Math. 68(1946), no. 2, 257–262 [Lévy–Marcinkiewicz theorem, p. 257–258].

[Ku49] K. Kunisawa, *On an analytical method in the theory of independent random variables*, Ann. Inst. Statist. Math., Tokyo 1(1949), 1–77 [6.3.2. Lévy–Marcinkiewicz theorem, p. 75].

[Le31] P. Lévy, *Sur les séries doint les termes sont des variables éventuelles indépendaintes*, Studia Math. 3(1931), 117–155.

[Li56] M. Lipschutz, *On strong bounds for sums of independent random variables which tend to a stable distribution*, Trans. Amer. Math. Soc. 8(1956), 135–154 [Marcinkiewicz and Lévy result, p. 136].

4.2.4. *Marcinkiewicz theorem on characteristic function* (*1938*). Let X be a real-valued random variable on a probabaility space (Ω, Σ, P) with the distribution function $F(x)$. The *characteristic function* (or the Fourier–Stieltjes transform) of the random variable X (or of the distribution function F) is a function $f\colon \mathbb{R} \to \mathbb{C}$ defined by

$$f(t) = E(e^{itX}) = \int_{\mathbb{R}} e^{itx} dF(x), t \in \mathbb{R}.$$

There is interest to decide whether a given function $f(t)$ can be a characteristic function, i.e., whether it admits the above representation. Necessary and sufficient conditions are known which a complex-valued function of a real variable t must satisfy in order to be a characteristic function of some random variable. The central result here is Bochner's theorem (1932), although its usefulness is limited because the main condition of the theorem, positive definiteness, is very hard to verify. Other theorems also exist, such as Mathias (1923), Khinchine (1937), or Cramér (1937), although their application is just as difficult (cf. Linnik [Li64], pp. 41–48). Pólya's theorem (1949), on the other hand, provides a very simple convexity condition which is sufficient but not necessary. However, these general conditions are not easily applicable. Therefore various conditions were derived which are restricted to certain classes of functions but are applied more readily.

A characteristic function f is said to be an *analytic characteristic function* if it coincides in $-\delta < t < \delta$ for some $\delta > 0$ with a function of complex variable $z = t + iv$ which is analytic in the disc $|z| < \delta$.

Marcinkiewicz, in the paper [M39a] from 1939, showed that the exponential function with the base e and exponent given by polynomial of degree higher than 2 is not a characteristic function of any random variable, that is, if $\varphi(t) = \exp(P(t))$, where P is a polynomial such that $P(0) = 0$ is a characteristic function of some random variable X, then $P(t)$ is a polynomial of degree 2 and X is a Gaussian variable.

Theorem 12 (Marcinkiewicz theorem on characteristic function 1938). *If a polynomial $P(t) = P_n(t) = \sum_{k=1}^{n} a_k\, t^k$ is of degree $n > 2$, then the function $\varphi(t) = \exp(P(t))$ is not a characteristic function. More general, any entire function of finite order ρ, which convergence exponent is smaller than ρ cannot be a characteristic function.*

In other words Marcinkiewicz's theorem asserts: No function of the form $\exp(\sum_{k=1}^{n} a_k z^k)$ with $n > 2$ can be a characteristic function; also if the function $\varphi(t) = \exp[P(t)]$ with $P(t) = \sum_{k=1}^{n} a_k t^k, a_k \in \mathbb{C}$, is a characteristic function, then either $P(t) = -a\, t^2 + ib\, t, a > 0, b \in \mathbb{C}$ (Gaussian law) or $P(t) = ib\, t$ (degenerate law).

Marcinkiewicz's theorem was extended to iterated exponents and certain functions of the form $f(t) = g(t) \exp[P(t)]$ by Lukacs [Lu58]: If

$$e_1(z) = \exp(z), e_2(z) = e^{e_1(z)}, \ldots, e_k(z) = e^{e_{k-1}(z)}$$

and $P_m(t) = \sum_{k=0}^{m} c_k t^k$ is a polynomial of degree $m > 2$, then for any $n \geq 1$ the function $f_n(t) = c_n\, e_n[P_m t)]$ with constant c_n determined by the condition that $f_n(0) = 1$ can not be a characteristic function (for $n = 1$ this is the Marcinkiewicz theorem).

Further extension was done by Christensen [Ch62] to certain functions of the form $f_n(t) = c_n\, g(t)\, e_n[P_m(t)]$, where $g(t)$ is some specified characteristic function. Cairoli [Ca64] investigated similar problems for meromorphic functions of finite order. Miller

[Mi67] studied entire functions of the form $g(t)\{\exp[P(t)]\}$ or $f\{\exp[P(t)]\}$, where $g(t)$ and $f(t)$ are entire functions while $P(t)$ is a polynomial.

Important generalizations of Marcinkiewicz theorem were obtained by Ostrovskiĭ (1962, 1966, 1983). In 1962 Ostrovskiĭ [Os62] proved a conjecture of Linnik on strengthened Marcinkiewicz theorem for entire characteristic functions without zeros (simpler proof he gave in [Os83]): If a characteristic function $f(t)$ has the form $f(t) = \exp[g(t)]$, where $g(t)$ is entire function such that $\log^+ \max_{|z|=r} |f(z)| = o(r)$ as $r \to \infty$, then $f(t)$ is the characteristic function of the Gaussian law.

Marcinkiewicz theorem was further generalized by Sapogov (1976, 1979), Kamynin (1979), Golinskiĭ (1986, 1988) and Feldman (1989) on other classes of analytic functions, and also on distribution functions of several variables by Rajagopal and Sudarshan (1974) as well as on matrix-valued analytic characteristic functions by Gyires (1983).

Marcinkiewicz theorem is useful and used by many authors in studies concerning the characterization of the normal distribution. For example, we can find it in the books by Linnik (1964), Lukacs and Laha (1964), Ramachandran (1967), Lukacs (1970), Kagan, Linnik and Rao (1973), Linnik and Ostrovskiĭ (1977), Bryc (1995) and Feldman (1990, 2008).

Among the books and papers thematically related to Theorem 12 it is worth to mention the following ones:

[Li64] Yu. V. Linnik, *Decomposition of Probability Distributions*, Oliver & Boyd, Edinburgh-London 1964 [3.3.1. Marcinkiewicz theorem, pp. 56–58]; Russian version in Izdat. Leningrad. Univ., Leningrad 1960.

[LL64] E. Lukacs and R. G. Laha, *Applications of Characteristic Functions*, Hafner, New York 1964 [Marcinkiewicz' theorem (Lemma 5.1.2), p. 75].

[Ra67] B. Ramachandran, *Advanced Theory of Characteristic Functions*, Statistical Publishing Soc., Calcutta 1967 [3.13 and 3.14. Marcinkiewicz theorems, pp. 63–64].

[Lu70] E. Lukacs, *Characteristic Functions*, 2nd ed., Hafner, New York 1970 [Corollary to theorem 7.3.3 (Theorem of Marcinkiewicz), p. 213; Theorem 7.3.4 – theorem of Marcinkiewicz, pp. 221–225].

[KLR73] A. M. Kagan, Yu. V. Linnik and C. R. Rao, *Characterization Problems in Mathematical Statistics*, Wiley, New York-London-Sydney 1973 [Lemma 1.4.2. Marcinkiewicz' theorem, p. 25]; Russian version in Nauka, Moscow 1972.

[LO77] Ju. V. Linnik and I. V. Ostrovskiĭ, *Decomposition of Random Variables and Vectors*, AMS, Providence 1977 [II. 5. Marcinkiewicz's theorem, pp. 41–42 and 361]; Russian version in Nauka, Moscow 1972.

[Fe90] G. M. Feldman, *Arithmetic of Probability Distributions, and Characterization Problems on Abelian Groups*, Naukova Dumka, Kiev 1990 (Russian); English transl. in AMS, Providence 1993 [Appendix 1. Group analogs of the Marcinkiewicz theorem and the Lukacs theorem, pp. 173–177].

[Br95] W. Bryc, *The normal distribution. Characterizations with applications*, Lecture Notes in Statistics 100, Springer, New York 1995 [in Section 2.5 two classical theorems appeared, Cramér decomposition theorem and Marcinkiewicz theorem, giving criteria for normality].

[Fe08] G. Feldman, *Functional Equations and Characterization Problems on Locally Compact Abelian Groups*, EMS Tracts in Math. 5, European Math. Soc., Zürich 2008 [II. 5. Polynomials on locally compact Abelian groups and the Marcinkiewicz theorem, pp. 38–49].

[Ca64] R. Cairoli, *Sur les fonctions caractéristiques de lois de probabilité*, Publ. Inst. Statist. Univ. Paris 13(1964), 45–53.

[Ch62] I. F. Christensen, *Some further extensions of a theorem of Marcinkiewicz*, Pacific J. Math. 12(1962), 59–67.

[Fe89] G. M. Feldman, *Marcinkiewicz and Lukacs theorems on abelian groups*, Teor. Veroyatnost. i Primenen. 34(1989), no. 2, 330–339; English transl. in Theory Probab. Appl. 34(1989), no. 2, 290–297.

[Go86] L. B. Golinskiĭ, *An estimate for stability in the Marcinkiewicz theorem for fourth-degree polynomials*, in: "Mathematical Physics, Functional Analysis", Naukova Dumka, Kiev 1986, 118–126 (Russian).

[Go88] L. B. Golinskiĭ, *Stability estimates in a theorem of J. Marcinkiewicz*, in: "Stability Problems for Stochastic Models", Proc. Internat. Seminar (Sukhumi, October 1987), Edited by V. M. Zolotarev and V. V. Kalashnikov, Moscow 1988, 8–24; English transl. in J. Soviet Math. 57(1991), no. 4, 3193–3209.

[Gy83] B. Gyires, *On matrix-valued analytic characteristic functions*, Publ. Math. Debrecen 30(1983), no. 1–2, 133–142.

[Ka79] I. P. Kamynin, *Generalization of the theorem of Marcinkiewicz on entire characteristic functions of probability distributions*, Zap. Nauch. Sem. Leningrad. Otd. Mat. Inst. Steklov. (LOMI) 85(1979), 94–103; English transl. in J. Math. Sci. 20(1982), no. 3, 2175–2180.

[Lu58] E. Lukacs, *Some extensions of a theorem of Marcinkiewicz*, Pacific J. Math. 8(1958), 487–501.

[Lu72] E. Lukacs, *A survey of the theory of characteristic functions*, Advances in Appl. Probability 4(1972), 1–38 [Theorem 3.6 (Marcinkiewicz), p. 14].

[Mi67] H. D. Miller, *Generalization of a theorem of Marcinkiewicz*, Pacific J. Math. 20(1967), 261–274.

[Os62] I. V. Ostrovskiĭ, *Application of a rule of Wiman and Valiron to the study of the characteristic functions of probability laws*, Dokl. Akad. Nauk SSSR 143(1962), 532–535 (Russian).

[Os66] I. V. Ostrovskiĭ, *On the growth of entire characteristic functions of probabilistic laws*, in: "Contemporary Problems in Theory Anal. Functions" (Internat. Conf., Erevan 1965), Izdat. Nauka, Moscow 1966, 239–245 (Russian).

[Os83] I. V. Ostrovskiĭ, *On the growth of entire characteristic functions*, Lecture Notes in Math. 982(1983), 151–155.

[RS74] A. K. Rajagopal and E. C. G. Sudarshan, *Some generalizations of the Marcinkiewicz theorem and its implications to certain approximation schemes in many-particle physics*, Phys. Rev. A (3) 10(1974), 1852–1857.

[Sa76] N. A. Sapogov, *Stability for the Marcinkiewicz theorem. The case of a fourth degree polynomial*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 61(1976), 107–124, 137–138 (Russian).

[Sa79a] N. A. Sapogov, *Weak stability of J. Marcinkiewicz's theorem and some inequalities for characteristic functions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 85(1979), 193–196 (Russian).

[Sa79b] N. A. Sapogov, *The problem of stability for J. Marcinkiewicz's theorem*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 87(1979), 104–124, 208–209 (Russian).

4.3. Real mathematical analysis. In this part we will deal with Marcinkiewicz integral, Marcinkiewicz function, theorem on strong differentiation of integrals, maximal function, strong maximal function and Marcinkiewicz decomposition being a prototype of the Calderón–Zygmund decomposition.

4.3.1. *Marcinkiewicz integral, Marcinkiewicz function and decomposition of Marcinkiewicz–Zygmund.* Examining the existence of the conjugate function and its weak type $(1,1)$ Marcinkiewicz analyzed a special function and structure of closed subsets. Let us consider the one-dimensional case. For a closed set $P \subset \mathbb{R}^1$ and a point x let

$$\delta(x) = \delta(x, P) = \inf\{|x-y| : y \in P)\}$$

denote the distance of x to P. This function satisfies the Lipschitz condition, i.e., $|\delta(x) - \delta(y)| \leq |x-y|$. Marcinkiewicz proved the following result:

THEOREM 13 (Marcinkiewicz 1938). (i) *If P is a closed subset of a bounded open interval (a,b) and $\lambda > 0$, then the integral*

$$I_\lambda(x) = I_\lambda(x;P) = \int_a^b \frac{\delta(y)^\lambda}{|x-y|^{1+\lambda}}\,dy \tag{27}$$

is finite for almost all $x \in P$. Moreover, $I_\lambda \in L^1(P)$ and $\int_P I_\lambda(x)\,dx \;\; \leq \frac{2}{\lambda} m\left((a,b) \setminus P\right)$.

(ii) *If P is a closed subset in $\mathbb{R}^1$ and f a nonnegative integrable function on $\mathbb{R}^1 \setminus P$, then the function*

$$J_\lambda(f)(x) = \int_{\mathbb{R}^1} \frac{\delta(y)^\lambda f(y)}{|x-y|^{1+\lambda}}\,dy \tag{28}$$

is integrable on P, and hence is finite almost everywhere in P.

In the limiting case $\lambda = 0$, (27) and (28) should be replaced by the integrals

$$I_0(x) = \int_a^b \frac{(\log 1/\delta(y))^{-1}}{|x-y|}\,dy, J_0(f)(x) = \int_a^b \frac{f(y)\,(\log 1/\delta(y))^{-1}}{|x-y|}\,dy, \tag{29}$$

respectively. The integrals in (27), (28) and (29) are called *Marcinkiewicz integrals.* The integral (27) is discussed in the papers [M36a], [MZ36], [M38h], [M39f] and the integrals (29) in the paper [M35d]. Zygmund noted the following (cf. [Zy64], p. 5):

> *Marcinkiewicz proved this theorem in a somewhat different form by considering in* (27) *instead of the function* $\delta(x)$ *the function* $\psi(x)$ *which is equal to* 0 *in* P *and is equal to* d *in each interval contiguous to* P *and having length* d*, but the proofs in both cases are analogous, and the function* δ *is easier to use than* ψ*, especially if we consider the analogue of the theorem in the* n*-dimensional space. The proof of the theorem is not particularly difficult, and a discrete sum somewhat similar to the integral (27) in the case* $\lambda = 2$ *appears in an earlier paper of Besicovitch [Be26], in the proof of the existence of the conjugate function; it is possible that Marcinkiewicz knew that paper. The merit of Marcinkiewicz was that he understood the significance of the result transcending its individual application, and by using it systematically succeded in obtaining a number of very interesting results in the theory of trigonometric series.*

In general, in $\mathbb{R}^n$, for a closed subset P in $\mathbb{R}^n, \lambda > 0$ and f nonnegative measurable function on $\mathbb{R}^n$, we consider the *Marcinkiewicz integral*

$$J_\lambda(f)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\delta(\mathbf{y})^\lambda f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\lambda}}\, d\mathbf{y} (\mathbf{x} \in \mathbb{R}^n) \tag{30}$$

and their modified forms

$$H_\lambda(f)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\delta(\mathbf{y})^\lambda f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\lambda} + \delta(\mathbf{x})^{n+\lambda}}\, d\mathbf{y}, H'_\lambda(f)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\delta(\mathbf{y})^\lambda f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\lambda} + \delta(\mathbf{y})^{n+\lambda}}\, d\mathbf{y},$$

introduced by Carleson and Zygmund. Observe that

$$2^{-n-\lambda-1}\, H'_\lambda(f)(\mathbf{x}) \leq H_\lambda(f)(\mathbf{x}) \leq 2^{n+\lambda+1}\, H'_\lambda(f)(\mathbf{x}).$$

The proof of the following theorem is given in ([WZ77], Thm 9.19): *If $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$ and $\lambda > 0$, then $H_\lambda(f) \in L^p(\mathbb{R}^n)$ and*

$$\|H_\lambda(f)\|_p \leq C\, \|f\|_p,$$

where constant $C > 0$ is independent of f. In particular, $\|J_\lambda(f)\|_{L^p(P)} \leq C\, \|f\|_p$.

The Marcinkiewicz integral was an important tool in the proof of Calderón–Zygmund estimate of weak type $(1, 1)$ for n-dimensional strongly singular integrals (Hilbert integrals) (cf. Calderón and Zygmund [CZ52], and Stein [St75], pp. 14–19).

Different variants and generalizations of the Marcinkiewicz integral, and boundedness, not only in L^p-spaces, were studied by Ostrow and Stein [OS57], Yano [Ya59], Zygmund [69], Fefferman and Stein [FS71], Yano [Ya75], A. P. Calderón [Ca76], C. P. Calderón [Ca78], Kruglyak and Kuznetsov [KK07].

[Zy59] A. Zygmund, *Trigonometric Series, Vol. I, II.*, Cambridge Univ. Press, Cambridge 1959 [IV.2. A theorem of Marcinkiewicz, pp. 129–130].

[St75] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, New Jersey 1975 [2.3. Integral of Marcinkiewicz, pp. 14–15].

[WZ77] R. L. Wheeden and A. Zygmund, *Measure and Integral. An Introduction to Real Analysis*, Marcel Dekker, New York-Basel 1977 [Theorem of Marcinkiewicz, pp. 95–96; 4. The Marcinkiewicz integral, pp. 157–159].

[GR85] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam 1985 [Marcinkiewicz integrals, pp. 502–503 and 523].

[St93] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton 1993 [Marcinkiewicz integral, p. 76].

[GGKK] I. Genebashvili, A. Gogatishvili, V. Kokilashvili and M. Krbec, *Weight Theory for Integral Transforms on Spaces of Homogeneous Type*, Longman, Harlow 1998 [7.1. Weighted inequalities for the Marcinkiewicz integral, pp. 291–294]

[Be26] A. S. Besicovitch, *On a general metric property of summable functions*, J. London Math. Soc. 1(1926), 120–128.

[Ca76] A. P. Calderón, *On an integral of Marcinkiewicz*, Studia Math. 57(1976), no. 3, 279–284.

[Ca78] C. P. Calderón, *On a lemma of Marcinkiewicz*, Illinois J. Math. 22(1978), no. 1, 36–40.

[CZ52] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. 88(1952), 85–139.

[Ya59] S. Yano, *On a lemma of Marcinkiewicz and its applications to Fourier series*, Tohoku Math. J. (2) 11(1959), 191–215.

[Ca66] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. 116(1966), 135–157.

[FS71] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 93(1971), 107–115.

[FS72] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129(1972), no. 3–4, 137–193 [standard Marcinkiewicz "distance function integral", p. 190].

[KK07] N. Kruglyak and E. A. Kuznetsov, *The limiting case of the Marcinkiewicz integral: growth for convex sets*, Proc. Amer. Math. Soc. 135(2007), no. 10, 3283–3293.

[OS57] E. H. Ostrow and E. M. Stein, *A generalization of lemmas of Marcinkiewicz and Fine with applications to singular integrals*, Ann. Scuola Norm. Sup. Pisa (3) 11(1957), 117–135.

[Ya75] S. Yano, *On Marcinkiewicz integral*, Tohoku Math. J. (2) 27(1975), no. 3, 381–388.

[Zy69] A. Zygmund, *On certain lemmas of Marcinkiewicz and Carleson*, J. Approx. Theory 2(1969), 249–257.

The Marcinkiewicz function appeared in his paper [M38h] from 1938, which subject is on the borderline of real and complex variable. As is known Littlewood and Paley (1936) considering the behaviour on the boundary of analytic function $f(z)$ in the unit circle $|z| < 1$, which real part is $f(\theta)$, introduced the function of real variable

$$g(f)(\theta) = \left(\int_0^1 (1-r)|f'(re^{i\theta})|^2\, dr \right)^{1/2}$$

and proved that $\|g(f)\|_p = (\int_0^{2\pi} |g(f)(\theta)|^p\, d\theta)^{1/p} \approx \|f\|_p = (\int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta)^{1/p}$ for $1 < p < \infty$ if $\int_0^{2\pi} f(\theta)\, d\theta = 0$ (the equivalence constants depend only on p). Marcinkiewicz and Zygmund [MZ38c] showed also that $\|g(f)\|_p \le C_p \|f\|_p$ for $0 < p < \infty$. Luzin (1930) considered a function

$$s(f)(\theta) = \left(\iint_\Omega |f'|^2\, d\omega \right)^{1/2},$$

where Ω is a "triangle" area in $|z| < 1$ with a vertex at $e^{i\theta}$. Marcinkiewicz and Zygmund in [MZ38c] also obtained the following estimates:

$$\|s(f)\|_p \le A_p \|f\|_p \text{ for } 0 < p < \infty \text{ and } \|f\|_p \le B_p \|s(f)\|_p \text{ for } 1 < p < \infty.$$

Marcinkiewicz and some others before him were seeking for an analog of function g without use of complex variable. At the first moment we can think that if f is 2π-periodic function from L^2, then the expression

$$\left(\int_0^{2\pi} |f(\theta+t) - f(\theta-t)|^2 \frac{dt}{t} \right)^{1/2}$$

will be proper, but there are simple examples of continuous functions f indicating that this integral can be infinite for all θ. Marcinkiewicz accurately predicted the usefulness of the expression μ, now called *Marcinkiewicz function*, given by the formula

$$\mu(f)(x) := \left(\int_0^{2\pi} \left| \frac{F(\theta+t) + F(\theta-t) - 2F(\theta)}{t} \right|^2 \frac{dt}{t} \right)^{1/2}, \quad F(\theta) = \int_0^\theta f(t)\, dt + C, \quad (31)$$

about which he proved the following result ([Ma38g], Thm 1): *Let F be a 2π-periodic function from L^2. If F is differentiable at each point of the set of positive measure E, then the integral (31) is finite at almost each point of the set E.*

Marcinkiewicz considered in [M38h] for 2π-periodic functions from $L^p, p > 1$ a more general situation, namely, functions $\mu_r, r \geq 1$, defined as

$$\mu_r(f)(x) := \Big(\int_0^{2\pi} \Big|\frac{F(\theta+t)+F(\theta-t)-2F(\theta)}{t}\Big|^r \frac{dt}{t}\Big)^{1/r}$$

and proved the following estimates:

$$\|\mu_p(f)\|_p \leq B_p \|f\|_p, \text{ for } p \geq 2 \text{ and}$$
$$\|f\|_p \leq C_p \|\mu_p(f)\|_p \text{ for } 1 < p \leq 2 \text{ with } \int_0^{2\pi} f(t)\, dt = 0.$$

Marcinkiewicz raised the question if for $\mu = \mu_2$ we have

$$A_p \|f\|_p \leq \|\mu_2(f)\|_p \leq B_p \|f\|_p,$$

assuming, of course, in the first inequality that $\int_0^{2\pi} f(t)\, dt = 0$. Zygmund ([Zy44], Thm 1, p. 184) gave a positive solution. An analog of the Marcinkiewicz function on $\mathbb{R}$ is, for $f \in L^1$, the following function

$$\mu(f)(x) = \Big(\int_{\mathbb{R}^1} \Big|\frac{F(\theta+t)+F(\theta-t)-2F(\theta)}{t}\Big|^2 \frac{dt}{t}\Big)^{1/2} \tag{32}$$

and on $\mathbb{R}^n$

$$\mu(f)(x) = \Big(\int_{\mathbb{R}^n} \Big|\frac{F(\theta+t)+F(\theta-t)-2F(\theta)}{t}\Big|^2 \frac{dt}{|t|^n}\Big)^{1/2}$$

For the function (32) Waterman [Wa57] proved that if $f \in L^p(\mathbb{R}^1), p > 1$, then $\|\mu(f)\|_p \approx \|f\|_p$. Definitions of g-function, s-function and Marcinkiewicz function μ in $\mathbb{R}^n$ and all the above estimates in the case of n-variables were given by Stein [St58].

Investigations of the Marcinkiewicz function were and are still carried out in different directions, see e.g. [Wh69], [Wa72], [CW82], [CW83], [TW90], [SY99], [HMY07], and also [Zy44], [Ca50], [St58], [Zy59].

[Zy59] A. Zygmund, *Trigonometric Series, Vol. I, II.*, Cambridge Univ. Press, Cambridge 1959 [XIV.5. The Marcinkiewicz function $\mu(\theta)$, pp. 129–130].

[Ca50] A. P. Calderón, *On a theorem of Marcinkiewicz and Zygmund*, Trans. Amer. Math. Soc. 68(1950), 55–61.

[CW82] S. Chanillo and R. L. Wheeden, *Distribution function estimates for Marcinkiewicz integrals and differentiability*, Duke Math. J. 49(1982), no. 3, 517–619.

[CW83] S. Chanillo and R. L. Wheeden, *Relations between Peano derivatives and Marcinkiewicz integrals*, in: "Conference on Harmonic Analysis in Honor of Antoni Zygmund" held at the University of Chicago, Chicago, Ill., March 2328, 1981, Edited by W. Beckner, A. P. Calderón, R. Fefferman and P. W. Jones. Wadsworth, Belmont 1983, Vol. II, 508–525.

[HMY07] G. Hu, Y. Meng and D. Yang, *Estimates for Marcinkiewicz integrals in BMO and Campanato spaces*, Glasg. Math. J. 49(2007), no. 2, 167–187.

[LP36] J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series*, Proc. London Math. Soc. (2) 42(1936), 52–89.

[Lu30] N. Lusin, *Sur une propriété des fonctions à carré sommable*, Bull. Calcutta Math. Soc. 20(1930), 139–154.

[SY99] W. Sakamoto and K. Yabuta, *Boundedness of Marcinkiewicz functions*, Studia Math. 135(1999), no. 2, 103–142.

[St58] E. M. Stein, *On the functions of Littlewood–Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. 88(1958), 430–466.

[TW90] A. Torchinsky and S. L. Wang, *A note on the Marcinkiewicz integral*, Colloq. Math. 60/61(1990), no. 1, 235–243.

[Wa72] T. Walsh, *On the function of Marcinkiewicz*, Studia Math. 44(1972), 203–217.

[Wa59] D. Waterman, *On an integral of Marcinkiewicz*, Trans. Amer. Math. Soc. 91(1959), 129–138.

[Wh69] R. L. Wheeden, *Lebesgue and Lipschitz spaces and integrals of the Marcinkiewicz type*, Studia Math. 32(1969), 73–93.

[Ya04] K. Yabuta, *Existence and boundedness of g_λ^*-function and Marcinkiewicz functions on Campanato spaces*, Sci. Math. Jpn. 59(2004), no. 1, 93–112.

[Zy44] A. Zygmund, *On certain integrals*, Trans. Amer. Math. Soc. 55(1944), 170–204.

Marcinkiewicz and Zygmund in paper [MZ36], studying the trigonometric series, consider the so-called Riemann derivatives. A function f, defined in a neighbourhood of a point x_0, has at this point *k-th Riemann derivative* if there is a limit of the quotient $\lim_{h\to 0}\frac{\Delta_h^k f(x_0)}{(2h)^k} = D_k f(x_0)$, where the k-th difference is given by formula

$$\Delta_h^k f(x) = \sum_{i=0}^{k}(-1)^i\binom{k}{i} f(x+(k-2i)h) = f(x+kh) - \binom{k}{1} f(x+(k-2)h)$$
$$+ (-1)^2\binom{k}{2} f(x+(k-4)h) + \ldots + (-1)^k\binom{k}{k}(f(x-kh)$$

The special case $k = 2$ is the known *Schwarz derivative*. The main result in the paper ([MZ36], Thm 1) is the following:

Theorem 14 (Marcinkiewicz–Zygmund 1936). *If at each point x_0 of a set E of positive measure the quotient $\frac{\Delta_h^k f(x_0)}{(2h)^k}$ is bounded when $h \to 0$ (in particular, if the Riemann derivative $D_k f(x_0)$ exists), then f is k-times differentiable for almost all points from E.*

The method of proof showed that for a function f of the particular form

$$f(x+h) = \sum_{j=0}^{k-1} a_j(x)\frac{h^j}{k!} + O(h^k)(x \in E), \tag{33}$$

and for arbitrary $\varepsilon > 0$ there exists a set P and functions g, b such that

(i) P is a perfect set and $|E \setminus P| < \varepsilon$
(ii) $f(x) = g(x) + b(x)$, where g is of class C^k (k-th derivative is continuous) and $|b(x)| \le C\,\delta(x,P)^k$.

In other words, the function f satisfying (33) can be decomposed on "good part" $g(x)$ and a "bad part" $b(x)$, and the bad part $b(x)$ can be nonzero only on a small set $E \setminus P$ and is estimated by the use of Marcinkiewicz integral (27) with $\lambda = k$.

This method was an important tool of proofs in the forties. However, in 1952 Calderón and Zygmund presented their famous decomposition for functions of n-variables (see [St75], Thm 3.2) and from then this method of proof has become the leading one. Methods used in the theory of singular Calderón–Zygmund integrals both Marcinkiewicz decomposition and Marcinkiewicz interpolation theorem are particularly useful and still used in various versions and variants.

[As67] J. M. Ash, *Generalizations of the Riemann derivative*, Trans. Amer. Math. Soc. 126(1967), 181–199 [generalization of the Marcinkiewicz–Zygmund theorem].

[CZ52] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. 88(1952), 85–139.

[Dy01] E. M. Dynkin, *Methods of the theory of singular integrals: Hilbert transform and Calderón–Zygmund theory*, in: "Commutative Harmonic Analysis I", Encyclopaedia Math. Sci. 15, Springer, Berlin 1991, 167–259 [Marcinkiewicz interpolation theorem, p. 174; Marcinkiewicz integral, p. 240].

[FW94] H. Fejzić and C. E. Weil, *Repairing the proof of a classical differentiation result*, Real Anal. Exchange 19(1993/94), no. 2, 639–643 [Marcinkiewicz–Zygmund theorem].

4.3.2. *Differentiation of integrals and maximal functions.* The classical Lebesgue theorem (1923) tells that if $f \in L^1_{\text{loc}}(\mathbb{R}^1)$ and $F(x) = \int_0^x f(t)\,dt$, then $F'(x) = f(x)$ for almost all $x \in \mathbb{R}^1$ (shortly a.e.) , i.e.,

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x) \quad \text{a.e.}$$

or, equivalently,

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(y)\,dy = f(x) \quad \text{a.e.}$$

An n-dimensional version has a form: if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then

$$\lim_{|Q|\to 0} \frac{1}{|Q|} \int_Q f(y)\,dy = f(x) \quad \text{for almost all } x \in \mathbb{R}^n,$$

where Q denotes n-dimensional cube with sides parallel to the coordinate system. Even a stronger assertion is true

$$\lim_{|Q|\to 0} \frac{1}{|Q|} \int_Q |f(y) - f(x)|\,dy = 0 \quad \text{a.e. in } \mathbb{R}^n.$$

An important tool in the proof of Lebesgue (and theorems of singular integrals and convergence almost everywhere) is a maximal function of Hardy–Littlewood

$$Mf(x) := \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy,$$

where supremum is taken over all cubes Q in $\mathbb{R}^n$ containing the point x. Note that we can equivalently analyze the maximal function, where instead of cubes Q we take n-dimensional balls $B(x,r)$ with center at x and radius $r > 0$

$$M_b f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy.$$

Hardy and Littlewood [HL30] defined the maximal function M in the one-dimensional case and showed the boundedness in $L^p(\mathbb{R}^1)$ for $p > 1$ (they did not prove the weak type $(1,1)$ of M, which is a surprise). Next important step was done by F. Riesz (1932) who proved the following inequality

$$(Mf)^*(t) \le A\, f^{**}(t) = A\,\frac{1}{t}\int_0^t f^*(s)\,ds \quad \text{for any } t > 0,$$

where f^* denotes the decreasing rearrangement of $|f(x)|$. From here we are getting the weak type $(1,1)$ of the maximal function.

N. Wiener (1939) considered the maximal function Mf in the n-dimensional case and proved, with the help of the Vitali covering theorem, its important property i.e. the weak type $(1,1)$. It is necessary to mention here that the same year Marcinkiewicz and Zygmund also investigated two-dimensional maximal function (from which without difficulties we get the n-dimensional case for $n \ge 2$) and they proved its weak type $(1,1)$ ([MZ39b], Lemma 2, p. 551). This fact was noted only by Stein and Wainger ([SW78], p. 1245).

Theorem 15 (Wiener 1939, Marcinkiewicz–Zygmund 1939). *The maximal function M is of weak type $(1,1)$, that is, for any $f \in L^1(\mathbb{R}^n)$ and arbitrary $\lambda > 0$ the following inequality holds*

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \le \frac{B}{\lambda}\int_{\mathbb{R}^n} |f(x)|\,dx. \tag{34}$$

Wiener (1939) proved even a stronger version of the weak $(1,1)$ inequality

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \le \frac{B}{\lambda}\int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda/2\}} |f(x)|\,dx \quad \forall\,\lambda > 0, \tag{35}$$

and Stein [St68] showed in 1969 the following reverse weak $(1,1)$ inequality for the maximal function

$$\frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}} |f(x)|\,dx \le C\,|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \;\; \forall\,\lambda > 0.$$

The classical Theorem 15 can be found in many books on analysis or harmonic analysis. We can mention as examples the books from part 4.1.1: Stein [1970, pp. 5–11], Stein and Weiss [1971, pp. 55–56], Sadosky [1979, pp. 199–202], de Guzmán [1981, pp. 133–134], Jorsboe and Mejlbro [1982, pp. 10–11], Kashin and Saakyan [1984, pp. 444–445], García-Cuerva and Rubio de Francia [1985, pp. 144–145], Torchinsky [1986, pp. 77–78], Brudnyĭ and Krugljak [1991, pp. 79–80], Folland [1999, p. 96], Duoandikoetxea [2001, p. 31], Arias de Reyna [2002, pp. 4–6], DiBenedetto [2002, p. 378], Nikolski [Ni02, p. 31], Taylor [2006, pp. 140–141], Grafakos [2008, pp. 80–81], Linares and Ponce [2009, p. 34].

Futher studies of the Wiener and Stein inequalities (for arbitrary measures in $\mathbb{R}^n$) can be found, e.g., in the paper [AKMP].

Note that if we have the weak type $(1,1)$ of the maximal function, then it is not difficult to prove the Lebesgue theorem, but equivalence with the Lebesgue theorem is not so easy and it was proved by Stein in [St61].

Let us now discuss the classical theorem of Jessen, Marcinkiewicz and Zygmund on strong differentiation of integrals proved in 1935 in the paper [MJZ35]. Let rectangle P in $\mathbb{R}^n, n \ge 2$ means the product of n nonempty one-dimensional intervals and let $\delta(\mathrm{P})$ be its diameter. Let also $\mathrm{P}_0 \subset \mathbb{R}^n$ be a fixed rectangle, for example, $\mathrm{P}_0 = \mathbb{R}^n$ or $\mathrm{P}_0 = \mathrm{I}^n, \mathrm{I} = [0,1]$ and $f \in L^1(\mathrm{P}_0)$.

We say that the integral of the function f is *strongly differentiable* at the point $x \in P_0$ if the limit

$$\lim_{\delta(\mathrm{P})\to 0} \frac{1}{|\mathrm{P}|}\int_{\mathrm{P}} f(y)\,dy$$

exists and is finite, where $\mathrm{P} \subset \mathrm{P}_0$ is any rectangle containing x. This limit is called the *strong derivative* of the integral of a function f at point x.

Saks ([Sa33], pp. 231–232 and [Sa34]) and Buseman–Feller ([BF34], pp. 243–247) showed the existence of a function $f \in L^1(\mathrm{I}^2)$ which integral is nowhere strongly differentiable. Zygmund ([Zy34], Thm 1) however proved that for any function $f \in L^p(\mathrm{P}_0), p > 1$, the strong derivative of the integral of a function f exists almost everywhere and is equal to $f(x)$.

Jessen, Marcinkiewicz and Zygmund ([MJZ35], Thm 2) proved strong differentiability of the integral of any function $f \in L^1(\log^+ L)^{n-1}$.

Theorem 16 (Jessen–Marcinkiewicz–Zygmund 1935). *Let $\mathrm{P}_0 \subset \mathbb{R}^n$ be a fixed rectangle. If function f is measurable and $|f(x)|\,(\log^+ |f(x)|)^{n-1}$ is integrable on the rectangle P_0, then the strong derivative of the integral of the function f exists for almost all points in P_0 and is equal to $f(x)$.*

Jessen, Marcinkiewicz and Zygmund ([MJZ35], Thm 8) also demonstrated that the statement in Theorem 16 is in some sense the best possible. Namely, let $\Phi\colon [0,\infty) \to [0,\infty)$ be an increasing function vanishing only at zero and having the property

$$\liminf_{u\to\infty} \frac{\Phi(u)}{u} > 0.$$

Denote by $L^{\Phi}(\mathrm{I}^n)$ the Orlicz class of all functions f on I^n such that $\Phi(|f|) \in L^1(\mathrm{I}^n)$. The authors of that work showed that if every function $f \in L^{\Phi}(\mathrm{I}^n)$ has almost everywhere strongly differentiable integral, then $f(\log^+ |f|)^{n-1} \in L^1(\mathrm{I}^n)$. Thus the largest Orlicz class for which all functions have almost everywhere strongly differentiable integrals is the class L^{Φ_0} generated by the Orlicz function $\Phi_0(u) = u(\log^+ u)^{n-1}$!.

Paper [MJZ35] is cited quite often, and on the conference on "Development of mathematics 1900–1950" held in Luxembourg in 1992, this paper was listed among the most important ones published in the year 1935 (cf. [DEMP], p. 22).

The maximal function appropriate to the strong differentiation is the *strong maximal function*

$$M_S f(x) := \sup_{\mathrm{P}\ni x} \frac{1}{|\mathrm{P}|}\int_{\mathrm{P}} |f(y)|\,dy,$$

where supremum is taken over all rectangles P from $\mathbb{R}^n$ containing the point x.

Note that the function M_S is not of the weak type $(1,1)$. Namely, if we take as f, for example, the characteristic function of the unit ball, then for large coordinates $x_1, x_2, \ldots, x_n$ of a point x the quantity $M_S f(x)$ is of order $(|x_1| \cdot \ldots \cdot |x_n|)^{-1}$ and the inequality for weak type $(1,1)$ cannot be true. On the other hand, $M_S f(x)$ is pointwise bounded by the composition of one-dimensional maximal functions in each coordinate independently $M_S f(x) \le M_1(M_2(\ldots(M_n f(x))\ldots)$. Each of these maximal functions $M_k, 1 \le k \le n$, is bounded in $L^p, p > 1$, therefore the function M_S is bounded in $L^p(\mathbb{R}^n)$ for $1 < p \le \infty$.

To prove the Jessen–Marcinkiewicz–Zygmund theorem with the help of the strong maximal function we need a proper replacement of the inequality on week type $(1,1)$ by another estimate. Apropriate estimate was given by de Guzmán [Gu74]: *the strong maximal function satisfies for any* $\lambda > 0$ *the best possible inequality*

$$|\{x \in \mathbb{R}^n : M_S f(x) > \lambda\}| \le C_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\left(1 + \frac{|f(x)|}{\lambda}\right)^{n-1} dx, \tag{36}$$

i.e., $M_S\colon L(1 + (\log^+ L)^{n-1})(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ *is bounded.*

A geometrical proof, with the help of the corresponding covering lemma, was given by A. Córdoba and R. Fefferman [CF75]. This proof is repeated in appendix of the book [Gu75].

An inequality of type (36) for the strong maximal function M_s, where by definition the supremum is taken over all rectangles P contained in I^n, that is, the inequality

$$|\{x \in \mathrm{I}^n : M_s f(x) > 4\lambda\}| \le D_n \int_{\mathrm{I}^n} \frac{|f(x)|}{\lambda} \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1} dx, \tag{37}$$

was proved for $n = 2$ by Flett [Fl55] with $D_2 = 4$ and $1 + \log(\cdot)$ in place $\log^+(\cdot)$, and for $n \ge 2$ by Fava [Fa1972].

Books containing the problem of differentiation of integrals are, for example, the following ones:

[Sa33] S. Saks, *Théorie de l'integral*, Monografie Matematyczne 2, Warszawa 1933.

[Sa37] S. Saks, *Theory of the Integral*, 2nd rev. ed., Monografie Matematyczne 7, Warszawa-Lwów 1937.

[Gu75] M. de Guzmán, *Differentiation of Integrals in* $\mathbb{R}^n$, Lecture Notes in Math. 481, Springer 1975; Russian transl. in Mir, Moscow 1978.

[Gu81] M. de Guzmán, *Real Variable Methods in Fourier Analysis*, North-Holland, Amsterdam 1981.

There are books, where Theorem 16 is cited as the Jessen–Marcinkiewicz–Zygmund theorem:

[KK91] V. Kokilashvili and M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Scientific, NJ 1991 [Jessen, Marcinkiewicz and Zygmund result, p. 142].

[Ko07] A. Korenovskii, *Mean Oscillations and Equimeasurable Rearrangements of Functions*, Springer, Berlin and UMI, Bologna 2007 [Theorem 1.3 (Jessen, Marcinkiewicz and Zygmund), p. 5].

[St93] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton 1993 [pp. 76 and 661].

[Zh96] L. Zhizhiashvili, *Trigonometric Fourier Series and Their Conjugates*, Kluwer, Dordrecht 1996 [Jessen, Marcinkiewicz and Zygmund result, pp. 130, 149, 152, 188, 189, 279].

Various discussion on Theorem 16, different proofs and generalizations can be found in many papers. Below are presented some publications related to the above problems, and those in which Jessen–Marcinkiewicz–Zygmund theorem is mentioned: Saks (1935), Burkill (1951), Smith (1956), Zygmund (1967), Bruckner (1971), Fava (1972), de Guzmán (1974, 1976, 1986), Cordoba and Fefferman (1975), Strömberg (1977), Stein and Wainger (1978), Fava, Gatto and Gutiérez (1980), Bagby (1983), Soria (1986), Stokolos (1998, 2005, 2008), Kuchta, Morayne and Solecki (2001), Hagelstein (2004).

[AKMP] I. U. Asekritova, N. Ya. Krugljak, L. Maligranda and L. E. Persson, *Distribution and rearrangement estimates of the maximal function and interpolation*, Studia Math. 124(1997), no. 2, 107–132

[Ba83] R. Bagby, *A note on the strong maximal function*, Proc. Amer. Math. Soc. 88(1983), no. 4, 648–650.

[Ba83a] R. Bagby, *Maximal functions and rearrangements: some new proofs*, Indiana Univ. Math. J. 32(1983), no. 6, 879–891.

[Br71] A. M. Bruckner, *Differentiation of integrals*, Amer. Math. Monthly 78(1971), no. 9, 1–51.

[Bu51] J. C. Burkill, *On the differentiability of multiple integrals*, J . London Math. Soc. 26(1951), 244–249.

[BF34] H. Busemann and W. Feller, *Zur Differentiation der Lebesgueschen Integrale*, Fund. Math. 22(1934), 226–256.

[Ch95] L. Chevalier, *Fonction maximale forte et intégrale de Marcinkiewicz*, J. Anal. Math. 65(1995), 161–178.

[CF75] A. Cordoba and R. Fefferman, *A geometric proof of the strong maximal theorem*, Ann. of Math. 102(1975), no. 1, 95–100.

[Du07] J. Duoandikoetxea, *The Hardy–Littlewood maximal function and some of its variants*, in: "Advanced Courses of Mathematical Analysis. II", World Sci. Publ., Hackensack, NJ 2007, 37–56.

[Fa72] N. Fava, *Weak type inequalities for product operators*, Studia Math. 42(1972), 271–288.

[FGG80] N. A. Fava, E. A. Gatto and C. Gutiérrez, *On the strong maximal function and Zygmund's class* $L(\log^+ L)^n$, Studia Math. 69(1980), no. 2, 155–158.

[Fl55] T. M. Flett, *Some remarks on a maximal theorem of Hardy and Littlewood*, Quart. J. Math., Oxford Ser. (2) 6(1955), 275–282.

[Gu74] M. de Guzmán, *An inequality for the Hardy–Littlewood maximal operator with respect to a product of differentiation bases*, Studia Math. 49(1974), 185–194.

[Gu76] M. de Guzmán, *Differentiation of integrals in* $\mathbb{R}^n$, Lecture Notes in Math. 541(1976), 181–185.

[Gu86] M. de Guzmán, *The evolution of some ideas in the theory of differentiation of integrals*, in: "Aspects of Mathematics and its Applications", North-Holland, Amsterdam 1986, 377–385.

[Ha04] P. A. Hagelstein, *Córdoba–Fefferman collections in harmonic analysis*, Pacific J. Math. 216(2004), no. 1, 95–109.

[Ha04a] P. A. Hagelstein, *Rearrangements and the local integrability of maximal functions*, Pacific J. Math. 216(2004), no. 1, 111–126.

[HL30] G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math. 54(1930), 81–116.

[KMS01] M. Kuchta, M. Morayne and S. Solecki, *A martingale proof of the theorem by Jessen, Marcinkiewicz and Zygmund on strong differentiation of integrals*, Lecture Notes in Math. 1755(2001), 158–161.

[Le97] A. K. Lerner, *On estimates for strong maximal functions*, Izv. Vyssh. Uchebn. Zaved. Mat. 1997, no. 7, 36–48; English transl. in Russian Math. (Iz. VUZ) 41(1997), no. 7, 33–45.

[Sa34] S. Saks, *Remark on the differentiability of the Lebesgue indefinite integral*, Fund. Math. 22(1934), 257–261.

[Sa35] S. Saks, *On the strong derivatives of functions of intervals*, Fund Math. 25(1935), 235–252.

[Sm56] K. Smith, *A generalization of an inequality of Hardy and Littlewood*, Canad. J. Math. 8(1956), 157–170.

[So86] F. Soria, *Examples and counterexamples to a conjecture in the theory of differentiation of integrals*, Ann. of Math. 123(1986), no. 1, 1–9.

[St61] E. M. Stein, *On limits of sequences of operators*, Ann. of Math. 74(1961), 140–170.

[St69] E. M. Stein, *Note on the class* $L\log L$, Studia Math. 32(1969), 305–310.

[SW78] E.M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. 84(1978), no. 6, 1239–1295.

[St98] A. M. Stokolos, *On a problem of A. Zygmund*, Mat. Zametki 64(1998), no. 5, 749–762; English transl. in Math. Notes 64(1998), no. 5–6, 646–657.

[St05] A. Stokolos, *Zygmund's program: some partial solutions*, Ann. Inst. Fourier (Grenoble) 55(2005), no. 5, 1439–1453.

[St08] A. Stokolos, *Properties of the maximal operators associated with bases of rectangles in* $\mathbb{R}^3$, Proc. Edinb. Math. Soc. (2) 51(2008), no. 2, 489–494.

[St77] J.-O. Strömberg, *Weak estimates on maximal functions with rectangles in certain directions*, Ark. Mat. 15(1977), no. 2, 229–240.

[Wi39] N. Wiener, *The ergodic theorem*, Duke Math. J. 5(1939), 1–18.

[Zy34] A. Zygmund, *On the differentiability of multiple integrals*, Fund. Math. 23(1934), 143–149.

[Zy67] A. Zygmund, *A note on the differentiability of integrals*, Colloq. Math. 16(1967), 199–204.

4.3.3. *The Marcinkiewicz multiplier theorem and Marcinkiewicz sets.* Suppose there is a given Fourier series $f(x) \sim \sum c_n e^{inx}$ of a function $f \in L^p[0, 2\pi], p \geq 1$. We ask what conditions the sequence of numbers $(\lambda_n)_{n\in\mathbb{Z}}$ must satisfy that the series $\sum \lambda_n c_n e^{inx}$ is also the Fourier series of some function from $L^p[0, 2\pi]$. In other words, we consider the multiplier transformation T_λ defined by a sequence of numbers $\lambda = (\lambda_n)_{n\in\mathbb{Z}}$ by the formula

$$T_\lambda f \sim \sum \lambda_n c_n e^{inx} \quad \text{if} \quad f \sim \sum c_n e^{inx},$$

and ask under what assumptions on λ the operator T_λ is bounded in $L^p[0, 2\pi]$.

For $p = 2$ such a characterization is $\lambda \in l^\infty$, and for $p = 1$ the answer is also known (see [To86], p. 129). The above question for $1 < p < \infty, p \neq 2$ is much more difficult and still unsolved. A certain condition of the sequence of numbers $(\lambda_n)_{n\in\mathbb{Z}}$ which implies boundedness of the operator T_λ in $L^p[0, 2\pi]$ for $p > 1$ was given by Marcinkiewicz in the paper [M39f].

Theorem 17 (Marcinkiewicz multiplier theorem 1939). *Let $1 < p < \infty$. If a sequence $\lambda = (\lambda_n)_{n\in\mathbb{Z}}$ is bounded and sums of differences over dyadic blocks are bounded, that is,*

$$\sup_n \sum_{2^n \leq |k| < 2^{n+1}} |\lambda_k - \lambda_{k-1}| \leq M < \infty, \tag{38}$$

then the operator T_λ is bounded in $L^p[0, 2\pi]$ and $\|T_\lambda f\|_p \leq C(\sup_{n\in\mathbb{Z}} |\lambda_n| + M)\|f\|_p$ for $f \in L^p$.

We can define the space of multipliers M_p for $1 \leq p \leq \infty$, as the space of all seqences $(\lambda_n)_{n\in\mathbb{Z}}$ such that $\| \sum \lambda_k c_k(f) e^{ikx}\|_p \leq C\|f\|_p$ for any trigonometric polynomial f with a constant $C > 0$ independent of f. The infimum over all such C defines a norm and the space M_p becomes a Banach space, and Figa–Talamanca (1965) additionally proved that this is a dual space for $1 < p \leq 2$ and was even able to find its predual.

The following statements are true: $M_2 = l^\infty, M_p = M_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $M_p \subset M_q \subset l^\infty$ if $1 \leq p \leq q \leq 2$. Theorem 16 of Marcinkiewicz means that if a sequence λ is bounded and satisfies (38), then it belongs to M_p.

Much more important, because of applications, is the corresponding Marcinkiewicz theorem for multiple series. For simplicity, let us write it in the two-dimensional case. Consider the multiplier transformation T_λ given by a double sequence $\lambda = (\lambda_{nm})_{n,m\in\mathbb{N}}$ with the formula

$$T_\lambda f \sim \sum \lambda_{nm} c_{nm} e^{i(nx+my)} \quad \text{as far as } f \sim \sum c_{nm} e^{i(nx+my)}$$

and dyadic intervals $I_k = \{i \in \mathbb{Z} : 2^{k-1} \leq |i| < 2^k\}, J_l = \{j \in \mathbb{Z} : 2^{l-1} \leq |j| < 2^l\}$, and denote

$$\Delta_1 \lambda_{nm} = \lambda_{n+1,m} - \lambda_{n,m},\ \Delta_2 \lambda_{nm} = \lambda_{n,m+1} - \lambda_{n,m} \text{and} \Delta_{1,2} = \Delta_1 \cdot \Delta_2.$$

Theorem 17' (Marcinkiewicz multiplier theorem 1939). *Let $1 < p < \infty$. If for a double sequence $\lambda = (\lambda_{nm})_{n,m\in\mathbb{Z}}$ the following suprema are finite*

$$A = \sup_{n,m} |\lambda_{n,m}|,\ B_1 = \sup_{k,m} \sum_{n\in I_k} |\Delta_1 \lambda_{n,m}|,\ B_2 = \sup_{n,l} \sum_{m\in J_l} |\Delta_2 \lambda_{n,m}|$$

and

$$B_{1,2} = \sup_{k,l} \sum_{n\in I_k} \sum_{m\in J_l} |\Delta_{1,2} \lambda_{n,m}|,$$

then the operator T_λ is bounded in $L^p([0, 2\pi]^2)$ and $\|T_\lambda f\|_p \leq C(A + B_1 + B_2 + B_{1,2})\|f\|_p$ for $f \in L^p$.

As concrete examples of multipliers (λ_{nm}) Marcinkiewicz presented the following ones ([M37f], Thm 3):

$$\frac{m^2}{n^2 + m^2}, \frac{n^2}{n^2 + m^2}, \frac{|mn|}{n^2 + m^2},$$

and informed that in this way some problem posed by Schauder is solved.

The corresponding multiplier theorem can be formulated also for the Fourier transform $\mathcal{F}$ in $L^p(\mathbb{R}^n)$. For a bounded measurable function m on $\mathbb{R}^n$ we define an operator T_m as follows:

$$T_m f(x) = \mathcal{F}^{-1}[m(\cdot)\mathcal{F}f(\cdot)](x),\ f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{39}$$

A function m is said to be an *L^p-multiplier* if

$$\|T_m f\|_p \leq C_p \|f\|_p, \quad \text{for all } f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n).$$

In this case $T_m(\cdot)$ can be extended to $L^p(\mathbb{R}^n)$. The smallest constant C_p is the norm of this operator in $L^p(\mathbb{R}^n)$ and it is denoted by the symbol $\|T_m\|_p$. Note that $\|T_m\|_2 = \|m\|_\infty$ and if m is an L^p-multiplier, $1 < p < \infty$, then it also is an $L^{p'}$-multiplier and $\|T_m\|_{p'} = \|T_m\|_p$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Marcinkiewicz multiplier theorem for the Fourier transform has the following form (let us formulate, for simplicity, only the one-dimensional case):

THEOREM 17" (Marcinkiewicz multiplier theorem 1939). *Let $m\colon \mathbb{R} \to \mathbb{R}$ be a bounded finction of class C^1 on each dyadic set $(-2^{k+1}, -2^k) \cup (2^k, 2^{k+1})$ for $k \in \mathbb{Z}$. Assume that the derivative m' of the function m satisfies the condition*

$$\sup_{k\in\mathbb{Z}} \left(\int_{-2^{k+1}}^{-2^k} |m'(t)|\,dt + \int_{2^k}^{2^{k+1}} |m'(t)|\,dt \right) \leq A < \infty. \tag{40}$$

Then m is L^p-multiplier for all $1 < p < \infty$ and

$$\|T_m\|_p \leq C \max(p, \frac{1}{p-1})^6 (\|m\|_\infty + A).$$

The next known result on multipliers for Fourier integrals is the Hörmander–Mikhlin theorem. In 1956 Mikhlin [Mi56] proved Marcinkiewicz result on Fourier integrals, and in 1960 Hörmander, in the paper [Ho60], gave a further generalization and simplification of the proofs. This result is sometimes called Hörmander–Mikhlin multiplier theorem, which in the simplest one-dimensional case has form: *Let $m\colon \mathbb{R} \to \mathbb{C}$ be a bounded function on $\mathbb{R}\backslash\{0\}$ and satisfying either the Mikhlin condition $|xm'(x)| \leq A$ or weaker the Hörmander condition*

$$\sup_{R>0} R \int_{R<|x|<2R} |m'(x)|^2\,dx \leq A^2 < \infty. \tag{41}$$

Then m is L^p-multiplier for all $p \in (1, \infty)$ and

$$\|T_m\|_p \leq C \max(p, \frac{1}{p-1})(\|m\|_\infty + A).$$

Moreover, T_m is of weak type $(1, 1)$.

Observe that in the one-dimensional case Marcinkiewicz theorem $17''$ is stronger than the Hörmander–Mikhlin theorem, i.e., from the condition (41) follows the condition (40). But if we write these statements in higher dimensions, then the criteria of being multiplier in Marcinkiewicz theorem and Hörmander–Mikhlin theorem are not comparable (see [Gr08], pp. 361–370). In addition, the assumption in Marcinkiewicz theorem does not guarantee the weak type $(1, 1)$ of the mapping T_m (see Kislyakov [Ki88], p. 161).

Problems of Fourier multipliers with an extensive literature can be found in the books listed below, and further generalizations of Marcinkiewicz and Hörmander–Mikhlin theorem, either weakening assumptions about λ or m or considering multipliers from spaces L^p to L^q for $1 \leq p \leq q \leq \infty$, and also investigating multipliers for functions with values in Banach spaces, can be found in many works. Below some of them are cited.

The Fourier transform can be consider on groups. Let G be a compact Abelian group. A subset E of its dual group is called a *Marcinkiewicz set* if the multiplier $m = \chi_E$ is of weak type $(1, 1)$. The name "Marcinkiewicz set" as well as "quasi-Marcinkiewicz set" (and also a class of *Marcinkiewicz systems* and *quasi-Marcinkiewicz systems*) was introduced and studied by Kwapień and Pełczyński [KP80], and further information regarding these sets, together with examples we can find in the paper by Kislyakov [Ki01].

We note further that Marcinkiewicz in a joint paper with Kaczmarz [MK38] gave conditions under which the sequence of numbers $\lambda = (\lambda_n)$ is an $L^p - L^q$-multiplier, and also that expansions are with respect to any bounded orthonormal system on $[0, 1]$ which is complete in $L^1[0, 1]$.

[St70] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton 1970 [IV. 6. The Marcinkiewicz multiplier theorem, pp. 108–112].

[La71] R. Larsen, *An Introduction to the Theory of Multipliers*, Springer, New York 1971 [Marcinkiewicz, p. 12, 132].

[EG77] R. E. Edwards and G. I. Gaudry, *Littlewood–Paley and Multiplier Theory*, Springer, Berlin-New York 1977 [1.1.4. The weak Marcinkiewicz multiplier theorem, pp. 5–17; Chapter 8. Strong forms of Marcinkiewicz multiplier theorem and Littlewood–Paley theorem for $\mathbb{R}, \mathbb{T}$ and $\mathbb{Z}$, pp. 148–179].

[Ni77] S. M. Nikolskiĭ, *Approximation of Functions of Several Variables and Imbedding Theorems*, 2nd ed., Nauka, Moscow 1977 (Russian) [Marcinkiewicz theorem and Marcinkiewicz multipliers, pp. 57–64].

[GR85] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam 1985 [Theorem 5.130. Marcinkiewicz multiplier theorem, pp. 511–512].

[To86] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, Orlando 1986 [Marcinkiewicz multiplier theorem, pp. 326–327].

[St95] A. I. Stepanets, *Classification and Approximation of Periodic Functions*, Kluwer, Dordrecht 1995 [3. Multiplicators. Marcinkiewicz theorem, pp. 266–268].

[Du01] Z. Duoandikoetxea, *Fourier Analysis*, AMS, Providence 2001 [8.4. The Marcinkiewicz multiplier theorem, pp. 166–168].

[St05] A. I. Stepanets, *Methods of Approximation Theory*, VSP, Leiden 2005 [5. Marcinkiewicz theorem, pp. 448–450].

[Gr08] L. Grafakos, *Classical Fourier Analysis*, 2nd ed., Springer, New York 2008 [2.5. Two multiplier theorems, pp. 359–370]

[ABG93] N. Asmar, E. Berkson and T. A. Gillespie, *Spectral integration of Marcinkiewicz multipliers*, Canad. J. Math. 45(1993), no. 3, 470–482.

[Bo68] J. Bournaud, *Sur les multiplicateurs de $\mathcal{F}L^p(\mathbb{R})$*, C. R. Acad. Sci. Paris Sér. A-B 267(1968), A919–921.

[Ca88] A. Carbery, *Differentiation in lacunary directions and an extension of the Marcinkiewicz multiplier theorem*, Ann. Inst. Fourier (Grenoble) 38(1988), no. 1, 157–168.

[CS75] W. C. Connett and A. L. Schwartz, *Unifying multiplier theorems of Hörmander, Marcinkiewicz, and Michlin type*, Bull. Amer. Math. Soc. 81(1975), no. 3, 570–572.

[GT77] G. Gasper and W. Trebels, *Multiplier criteria of Marcinkiewicz type for Jacobi expansions*, Trans. Amer. Math. Soc. 231(1977), no. 1, 117–132.

[GK01] L. Grafakos and N. J. Kalton, *The Marcinkiewicz multiplier condition for bilinear operators*, Studia Math. 146(2001), no. 2, 115–156.

[Ho60] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104(1960), 93–140.

[Ki88] S. Kislyakov, *Fourier coefficients of continuous functions and a class of multipliers*, Ann. Inst. Fourier (Grenoble) 38(1988), no. 2, 147–183.

[Ki91] S. V. Kislyakov, *Classical themes of Fourier analysis*, in: "Commutative Harmonic Analysis I", Encyclopaedia Math. Sci. 15, Springer, Berlin 1991, 113–165.

[Ki01] S. V. Kislyakov, *Banach spaces and classical harmonic analysis*, in: "Handbook of the Geometry of Banach Spaces", Vol. I, North-Holland, Amsterdam 2001, 871–898.

[Ki09] S. V. Kislyakov, *Weak type* $(1,1)$ *in the strengthened Marcinkiewicz theorem*, Funktsional. Anal. i Prilozhen. 43(2009), no. 3, 89–92; English transl. in Funct. Anal. Appl. 43(2009), no. 3, 236–238.

[Ko67] V. Kokilashvili, *On extensions of some Marcinkiewicz and Littlewood–Paley theorems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 15(1967), 239–243.

[KP80] S. Kwapień and A. Pełczyński, *Absolutely summing operators and translation-invariant spaces of functions on compact abelian groups*, Math. Nachr. 94(1980), 303–340.

[LMR68] W. Littman, C. McCarthy and N. Rivière, *L^p-multiplier theorems*, Studia Math. 30(1968), 193–217.

[Mi56] S. G. Mikhlin, *On the multipliers of Fourier integrals*, Dokl. Akad. Nauk SSSR (N.S.) 109(1956), 701–703.

[TW01] T. Tao and J. Wright, *Endpoint multiplier theorems of Marcinkiewicz type*, Rev. Mat. Iberoamericana 17(2001), no. 3, 521–558.

[Ve03] A. Venni, *Marcinkiewicz and Mihlin multiplier theorems, and RR-boundedness*, in: "Evolution Equations: Applications to Physics, Industry, Life Sciences and Economics" (Levico Terme, 2000), Progr. Nonlinear Differential Equations Appl. 55, Birkhäuser, Basel 2003, 403–413.

4.3.4. *Convergence of Riemann sums and Marcinkiewicz–Salem conjecture (1940).* Marcinkiewicz, while staying in Paris, has written together with Raphaël Salem paper [MS40], published in 1940 concerning Riemann sums.

Let $\mathbb{T} = [0,1) = \mathbb{R}/\mathbb{Z}$ with normalized Lebesgue measure m. For a measurable function f on $\mathbb{T}$ and $n \in \mathbb{N}$ we define the n-th Riemann sum of f as

$$R_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right),\ x \in \mathbb{T}. \tag{42}$$

In particular, when $x = 0$, we have the usual Riemann sums

$$R_n f = \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right).$$

– If f is Riemann integrable on $\mathbb{T}$, then for any $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} R_n f(x) = \int_0^1 f(t)\, dm. \tag{43}$$

– If f is only Lebesgue integrable on $\mathbb{T}$, then

$$\lim_{n\to\infty} \|R_n f(\cdot) - \int_0^1 f(t)\, dm\|_{L^1(\mathbb{T})} = 0. \tag{44}$$

Then it is natural to pose the question on pointwise convergence of these sums. The first investigations were done by Hahn (1914) who wanted to approximate Lebesgue integrals by the Riemann integrals. However, the first result is due to Jessen (1934):

– *if $f \in L^1(\mathbb{T})$ and if (n_k) is an increasing sequence of natural numbers in which next term divides previous one, then*

$$\lim_{k\to\infty} R_{n_k} f(x) = \int_0^1 f(t)\, dm \quad \textit{for almost all } x.$$

REMARK 2. (Marcinkiewicz–Salem 1940). The Jessen result is in some sense the best one, i.e., for the sequence $(2^n)_{n\geq 1}$ and for any positive increasing function w satisfying

$\lim_{x\to\infty} \frac{w(x)}{\ln x} = 0$ we can find a function f such that

$$\int_{\mathbb{T}} |f|w(|f|)dm < \infty \quad \text{and} \quad \int_{\mathbb{T}} \sup_{k\geq 0} |R_{2^k} f| dm = +\infty.$$

- Ursell (1937) and Marcinkiewicz–Salem (1940) showed the existence of a function $f \in L^1(\mathbb{T})$ such that $\limsup_{n\to\infty} |R_n f(x)| = +\infty$ for any x.
- Rudin (1964) showed even more, i.e., the existence of a function $f \in L^1(\mathbb{T})$ such that $\limsup_{n\to\infty} R_{2n+1} f(x) = +\infty$.

¿From this follows that we cannot have convergence almost everywhere even for bounded functions.

Theorem 18 (Marcinkiewicz–Salem 1940). (a) *If*

$$\int_{\mathbb{T}} [f(x+t) - f(x)]^2 \, dx = O(t^{\varepsilon}), \; \varepsilon > 0, \tag{45}$$

then the sequence $(R_n f)_{n\geq 1}$ is convergent almost everywhere to $\int_{\mathbb{T}} f dm$.

(b) *If*

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{[f(x+t) - f(x)]^2}{t|\ln \frac{t}{2}|} dt dx < \infty, \tag{46}$$

then the sequence of averages $\{A_n f = \frac{1}{n}\sum_{k=1}^n R_k f\}_{n\geq 1}$ is convergent almost everywhere to $\int_{\mathbb{T}} f dm$.

(c) *If*

$$\int_{\mathbb{T}} |f(x+t) - f(x)| \, dx = O\left(\frac{1}{|\ln t|^p}\right), \; p > 1, \tag{47}$$

then the sequence of averages $(A_n f)_{n\geq 1}$ is convergent almost everywhere to $\int_{\mathbb{T}} f dm$.

Note that condition (46) holds when for example

$$\int_{\mathbb{T}} [f(x+t) - f(x)]^2 \, dx = O(\frac{1}{\ln^2 |\ln t|}),$$

which is essentially weaker than (45). Moreover, if f is non-decreasing and $\int_{\mathbb{T}} |f(x)|^q dx < \infty$ for some $q > 1$, then (47) holds.

It is time to formulate a famous conjecture, namely the following one:

Marcinkiewicz–Salem conjecture (1940). *If $f \in L^2(\mathbb{T})$, then the sequence of averages $(A_n f = \frac{1}{n}\sum_{k=1}^n R_k f)_{n\geq 1}$ is convergent almost everywhere.*

Let us mention the resultat of Bourgain (1990) connected with this hypothesis: *If $f \in L^2(\mathbb{T})$, then the sequence $(R_n f)$ has logarithmic density, i.e.,*

$$\frac{1}{\ln N} \sum_{n=1}^{N} \frac{1}{n} R_n f \to \int_{\mathbb{T}} f dm a.e.$$

More information, proofs, connection with number theory and generalizations related to convergence almost everywhere can be found in the Rauch–Weber (2006) paper and other papers mentioned below:

[Bo90] J. Bourgain, *Problems of almost everywhere convergence related to harmonic analysis and number theory*, Israel J. Math. 71(1990), no. 1, 97–127.

[Je34] B. Jessen, *On the approximation of Lebesgue integrals by Riemann sums*, Ann. of Math. 35(1934), no. 2, 248–251.

[RW06] J.-J. Ruch and M. Weber, *On Riemann sums*, Note Mat. 26(2006), no. 2, 1–50.

[Ru64] W. Rudin, *An arithmetic property of Riemann sums*, Proc. Amer. Math. Soc. 15 (1964), 321–324.

[We04] M. A. Weber, *A theorem related to Marcinkiewicz–Salem conjecture*, Results Math. 45(2004), no. 1–2, 169–184.

[We05] M. Weber, *Almost sure convergence and square functions of averages of Riemann sums*, Results Math. 47(2005), no. 3–4, 340–354.

4.3.5. *Marcinkiewicz theorem on universal primitive functions (1935).* Luzin (1915) showed that every Lebesgue measurable and almost every finite function on an interval $[a, b]$ is almost everywhere derivative of a continuous function (a proof can be found in the Saks book [Sa37], pp. 217–218). Marcinkiewicz (1935) generalized this theorem in [M35a] by proving the following remarkable fact: there is a continuous function F, which is a generalized primitive function (antiderivative) for every a.e. finite Lebesgue measurable function f, that is, F is a universal generalized primitive function. Marcinkiewicz showed also that most functions are universal primitive functions, since in the class of continuous functions the functions which are not universal generalized primitive functions form a set of the first category in $C[a, b]$. Marcinkiewicz not only proved the existence of universal primitive function, but he also the first to use the word "universal" in such a context and the first to show that a set of universal elements is residual.

Theorem 19 (Marcinkiewicz theorem on universal primitive functions 1935). *Let $[a, b] \subset \mathbb{R}$ and let $(h_n)_{n=1}^{\infty}$ be a fixed sequence of nonzero real numbers converging to zero. Then there exists a continuous function $F\colon [a, b] \to \mathbb{R}$ having the following property: if $f\colon [a, b] \to \mathbb{R}$ is any Lebesgue measurable function, then there is a subsequence (h_{n_k}) of $(h_n)_{n=1}^{\infty}$ such that*

$$\lim_{k\to\infty} \frac{F(x + h_{n_k}) - F(x)}{h_{n_k}} = f(x) \quad \text{almost everywhere on } [a, b].$$

Such functions F constitute a residual set in $C[a, b]$.

Note that one and the same function F works for all functions f. Of course the subsequence depends on f. The function F may be called *generalized primitive function* (*antiderivative*) of f with respect to the given sequence $(h_n)_{n=1}^{\infty}$ and it is clear that such an F may be a generalized primitive function of many functions not equivalent to f.

Marcinkiewicz Theorem 19 with proof can be found in the books by Saks (1937), Bruckner (1978), Stromberg (1981) and Wise-Hall [WH93]:

[Sa37] S. Saks, *Theory of the Integral*, Monografie Matematyczne 7, Warszawa-Lwów 1937 [Marcinkiewicz theorem, p. 218].

[Br78] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Math. 659, Springer, Berlin-New York 1978 [Theorem 3.3. Marcinkiewicz, pp. 82–83].

[St81] K. R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth International, Belmont 1981 [Marcinkiewicz theorem with the proof, pp. 316–317].

[WH93] G. L. Wise and E. B. Hall, *Counterexamples in Probability and Real Analysis*, Oxford Univ. Press, New York 1993 [Marcinkiewicz theorem with the proof in Example 3.15, pp. 78–79].

Several authors have obtained strengthenings, generalizations and variants of Marcinkiewicz Theorem 19. Tuy [Tu59], [Tu60] shows that the values $f(x)$ can be derived numbers in a stronger sense, a martingale version is due to Lamb [La74], Aversa and Carrese [AC83] obtain an n-dimensional version for interval functions, Grande [Gr84] gives a Banach space-valued generalization, that is, for functions $f\colon [0,1] \to X$, where X is a Banach space. Cater [Ca89] replaces the difference quotient $(F(x+h) - F(x))/h$ by certain higher-order difference quotoients, and Gan and Stromberg [GS94] obtain the generalization of Marcinkiewicz's theorem for functions $f\colon [0,1]^n \to \mathbb{R}^n$. Smooth universal Marcinkiewicz functions were constructed by Krotov in [Kr91].

Joó [Jo89] studied the problem when one replaces a.e. convergence by convergence in $L^p[0,1]$ for any $0 < p < 1$. He (also Herzog and Lemmert [HL06]) showed the existence of a universal primitive F in the space $C[0,1]$ such that to each function $f \in L^p[0,1]$ there is a subsequence of $(F(x+\lambda_n) - F(x))/\lambda_n$ with limit f in $L^p[0,1]$. Several authors showed that one cannot choose here $p \geq 1$ (see Bogmér–Sövegjártó [BS87], Buczolich [Bu87] and Horváth [Ho87]). Herzog and Lemmert [HL09] proved a universality theorem from which we can deduce that F may be chosen to be Hölder continuous for each exponent $\alpha \in (0,1)$. Of course there are no Lipschitz continuous universal primitives since each Lipschitz continuous function is differentiable almost everywhere.

The general definition of the universality was given by Grosse-Erdmann in 1999. He presented this concept with several examples and references in a survey article [GE99]. Some related problems are raised in Laurinčikas [La03].

[AC83] V. Aversa and R. Carrese, *A universal primitive for functions of many variables*, Rend. Circ. Mat. Palermo (2) 32(1983), no. 1, 131–138 (Italian).

[BS87] A. Bogmér and A. Sövegjártó, *On universal functions*, Acta Math. Hungar. 49(1987), no. 1–2, 237–239.

[BL66] A. M. Bruckner and J. L. Leonard, *Derivatives*, Amer. Math. Monthly 73(1966), No. 4, 24–56 [Marcinkiewicz result, p. 29].

[Bu87] Z. Buczolich, *On universal functions and series*, Acta Math. Hungar. 49(1987), no. 3–4, 403–414.

[Ca89] F. S. Cater, *Some higher-dimensional Marcinkiewicz theorems*, Real Anal. Exchange 15(1989/90), no. 1, 269–274.

[GS94] X.-X. Gan and K. R. Stromberg, *On universal primitive functions*, Proc. Amer. Math. Soc. 121(1994), no. 1, 151–161.

[Gr84] E. Grande, *Sur un théoréme de Marcinkiewicz*, Problemy Mat. 4(1984), 35–41.

[GE99] K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. (N.S.) 36(1999), no. 3, 345–381 [Theorem 12 (Marcinkiewicz), p. 362].

[HL06] G. Herzog and R. Lemmert, *Universality of methods approximating the derivative*, Bull. Austral. Math. Soc. 73(2006), no. 3, 405–411.

[HL09] G. Herzog and R. Lemmert, *On Hölder continuous universal primitives*, Bull. Korean Math. Soc. 46(2009), no. 2, 359–365.

[Ho87] M. Horváth, *On multidimensional universal functions*, Studia Sci. Math. Hungar. 22(1987), no. 1–4, 75–78.

[Jo89] I. Joó, *On the divergence of eigenfunction expansions*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 32(1989), 3–36 (1990).

[Kr91] V. G. Krotov, *On the smoothness of universal Marcinkiewicz functions and universal* *trigonometric series*, Izv. Vyssh. Uchebn. Zaved. Mat. 1991, no. 8, 26–31; English transl. in Soviet Math. (Iz. VUZ) 35(1991), no. 8, 24–28.

[La74] C. W. Lamb, *Representation of functions as limits of martingales*, Trans. Amer. Math. Soc. 188(1974), 395–405.

[La03] A. Laurinčikas, *The universality of zeta-functions*, Acta Appl. Math. 78(2003), no. 1–3, 251–271.

[Tu59] H. Tuy, *The structure of measurable functions*, Dokl. Akad. Nauk SSSR 126(1959), 37–40 (Russian).

[Tu60] H. Tuy, *The "universal primitive" of J. Marcinkiewicz*, Izv. Akad. Nauk SSSR. Ser. Mat. 24(1960), 617–628 (Russian).

4.3.6. *Marcinkiewicz theorem on Perron integral and Marcinkiewicz–Zygmund integral.* If a function $f\colon [a,b] \to \mathbb{R}$ has derivative f' which is Riemann integrable, then from the fundamental theorem of calculus we have the equality

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{48}$$

The equality (48) is not always true, even if f is differentiable on $[a,b]$, since the derivative can be unbounded and thus not Riemann integrable on $[a,b]$. We would like to have such an integral, which has sense for all derivatives and which ensure the equality (48). For example for the function $f(x) = x^2 \cos \frac{\pi}{x^2}$ for $0 < x \le 1$ and $f(0) = 0$ it yields

$$\int_{I_n} |f'(x)|\,dx = \frac{1}{2n}, \text{ where } I_n = [(\frac{2}{4n+1})^{1/2}, \frac{1}{(2n)^{1/2}}] \text{ and, hence, } \int_0^1 |f'(x)|\,dx = \infty.$$

Thus f' is not Lebesgue integrable on $[0,1]$, nevertheless the improper integral

$$\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 f'(x)\,dx$$

exists. Consequently, the Lebesgue integral does not solve the problem.

Denjoy (1912) introduced an integral having the required property. The narrow Denjoy integral of function f is defined by existence of a continuous indefinite integral F on $[a,b]$ such that $F' = f$ almost everywhere in $[a,b]$ with some technical conditions. This integral is equivalent with the Perron integral defined in 1914 with the help of major and minor function of f. For the Perron integral the formula (48) is true if the function f is differentiable on $[a,b]$.

Let $f\colon [a,b] \to \mathbb{R}$. Then function $M\colon [a,b] \to \mathbb{R}$ is called a *major* function of f if $M(a) = 0$ and $\underline{D}M(x) \ge f(x)$ for $x \in [a,b]$, and $\underline{D}M(x) = \liminf_{h\to 0} \frac{M(x+h)-M(x)}{h}$, and function $m\colon [a,b] \to \mathbb{R}$ is called a *minor* function of f if $m(a) = 0$ and $\overline{D}m(x) \le f(x)$ for $x \in [a,b]$, where $\overline{D}m(x) = \limsup_{h\to 0} \frac{m(x+h)-m(x)}{h}$

A function $f\colon [a,b] \to \mathbb{R}$ is said to be *integrable in the sense of Perron* (P-integrable) on $[a,b]$ if there exist major function M of f and minor function m of f, and $\inf_M M(b) = \sup_m m(b) = I$. Their common value I is called the Perron integral of f on $[a,b]$ and is denoted by $(P)\int_a^b f(x)\,dx$. If f is P-integrable on $[a,b]$, then $F(u) = (P)\int_a^u f(x)\,dx, u \in [a,b]$, is continuous and $F' = f$ almost everywhere on $[a,b]$, and hence we are getting that f is Lebesgue measurable (F need not be absolutely continuous). Moreover, if f is P-integrable on $[a,b]$ and nonnegative, then f is also Lebesgue integrable on $[a,b]$.

One of the unexpected results on the Perron integral is Marcinkiewicz theorem contained in the book by Saks ([Sa37], p. 253) and not published in any paper of Marcinkiewicz (cf. Bullen [Bu90], p. 12).

THEOREM 20 (Marcinkiewicz theorem on Perron integral 1937). *A measurable function* $f\colon [a,b] \to \mathbb{R}$ *is Perron integrable if and only if it has one continuous major and one continuous minor function.*

This theorem was proved also by Tolstov [To39] and Denjoy [De49]. Generalizations of Theorem 20 on other Perron type integrals and on functions having possibly infinite values were formulated by McShane [Mc41], Frenkel and Cotlar [FC50], Sarkhel [Sa78]. Marcinkiewicz theorem is true for AP-integral (approximately continuous Perron integral), CP-integral (Cesàro–Perron integral) – see Bullen [Bu90]. Failure of the Marcinkiewicz theorem for SCP-integral (symmetric Cesàro–Perron integral) was proved by Sklyarenko [Sk99], for integrals defined by symmetric derivatives was proved by Skvortsov and Thomson [ST96] (see Thomson [Th94]), and for P_d-integral (dyadic Perron integral) was noticed by Skvortsov [Sk96]. Research, whether for a given integral Marcinkiewicz's theorem is true or not, continues to this day.

Marcinkiewicz theorem on existence of the Perron integral appeared in differential equations (in Peano theorem) – see e.g. Bullen and Vyborny [BV91].

In trigonometric series the fundamental problem was to define an integral in such a way that if a trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is convergent everywhere to a function $f(x)$, then $f(x)$ is integrable and the coefficients a_n and b_n are given by usual Fourier formulas. This problem was solved in many ways starting form Denjoy (1916).

Marcinkiewicz and Zygmund [MZ36] also gave such a way by defining the inverse of Borel derivative and using the Perron method. Now, a function M on $[a,b]$ is major of $f\colon [a,b] \to \mathbb{R}$, if $M(a) = 0$ and

$$\underline{B}_s M(x) = \liminf_{h\to 0} \frac{1}{h}\int_0^h \frac{M(x+t) - M(x-t)}{2t}\,dt \geq f(x),$$

and m on $[a,b]$ is minor of f, if $m(a) = 0$ and

$$\overline{B}_s m(x) \limsup_{h\to 0} \frac{1}{h}\int_0^h \frac{m(x+t) - m(x-t)}{2t}\,dt \leq f(x).$$

Marcinkiewicz–Zygmund integral (MZ-integral) is by definition

$$I = (MZ)\int_a^b f(x)\,dx := \inf_M M(b) = \sup_m m(b).$$

This integral has the required fundamental property: if the trigonometric series is convergent everywhere to $f(x)$, then f is integrable in the sense of Marcinkiewicz–Zygmund on $[0, 2\pi]$.

[Sa37] S. Saks, *Theory of the Integral*, Monografie Matematyczne 7, Warszawa-Lwów 1937 [Marcinkiewicz theorem, p. 253].

[Th94] B. S. Thomson, *Symmetric Properties of Real Functions*, Marcel Dekker, New York 1994 [7.61. Marcinkiewicz–Zygmund theorem, pp. 289–292].

[Go94] R. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, AMS, Providence 1994 [Theorem 8.20. A result due to Marcinkiewicz, p. 131].

[Bu90] P. S. Bullen, *Some applications of a theorem of Marcinkiewicz*, New Integrals (Coleraine, 1988), Lecture Notes in Math. 1419, Springer, Berlin 1990, 10–18.

[BV91] P. S. Bullen and R. Výborný, *Some applications of a theorem of Marcinkiewicz*, Canad. Math. Bull. 34(1991), no. 2, 165–174.

[De49] A. Denjoy, *Lecons sur le Calcul des Coefficients d'une Série Trigonométrique*, Vol. IV, Paris 1949.

[FC50] Y. Frenkel and M. Cotlar, *Mayorantes y minorantes no-aditivas en la teoria de la integral de Perron–Denjoy*, Revista Acad. Ci. Madrid 44(1950), 411–426.

[Je51] R. L. Jeffery, *Non-absolutely convergent integrals*, in: "Proc. Second Canadian Math. Congress" (Vancouver 1949), University of Toronto Press, Toronto 1951, 93–145 [Marcinkiewicz–Zygmund integral, pp. 138, 143].

[Ke38] S. Kempisty, *Sur les fonctions absolument semi-continues*, Fund. Math. 30(1938), 104–127 [Marcinkiewicz theorem on Perron integral, pp. 104 and 125].

[Mc42] E. J. McShane, *On Perron integration*, Bull. Amer. Math. Soc. 48(1942), 718–726.

[Sa78] D. N. Sarkhel, *A criterion for Perron integrability*, Proc. Amer. Math. Soc. 71(1978), no. 1, 109–112.

[Sk99] V. A. Sklyarenko, *On a property of the Burkill SCP-integral*, Mat. Zametki 65(1999), no. 4, 599–606; English transl. in Math. Notes 65(1999), no. 3–4, 500–505.

[Sk96] V. A. Skvorcov, *The Marcinkiewicz–Zygmund integral and its connection with the SCP-integral of Burkill*, Vestnik Moskov. Univ. Ser. I Mat. Meh. 27(1972), no. 5, 78–82 (Russian).

[Sk96] V. A. Skvortsov, *On the Marcinkiewicz theorem for the binary Perron integral*, Mat. Zametki, 59(1996), no. 2, 267–277; English transl. in Math. Notes, 59(1996), no. 2, 189–195.

[ST96] V. A. Skvortsov and B. S. Thomson, *Symmetric integrals do not have the Marcinkiewicz property*, Real Anal. Exchange 21(1995/96), no. 2, 510–520.

[To39] G. P. Tolstov, *Sur l'intégrale de Perron*, Mat. Sb. 5(1939), 647–660.

[VS71] I. A. Vinogradova and V. A. Skvortsov, *Generalized integrals and Fourier series*, Itogi Nauk. Mat. Anal. 1970, Akad. Nauk SSSR, Moscow 1971, 67–107; English transl. in J. Soviet Math. 1(1973), no. 6, 677–703.

4.4. Fourier series and orthogonal series. Consider a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{49}$$

or, formally equivalent, series in the complex form

$$\sum_{n=-\infty}^{\infty} c_n\, e^{inx}, \tag{50}$$

assuming $b_0 = 0$ we get $c_n = (a_n - ib_n)/2, c_{-n} = (a_n + ib_n)/2$.

A trigonometric series (49) is a Fourier series of a 2π-periodic function $f \in L^1[-\pi, \pi]$, if coefficients (Fourier coefficients) are given by formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx, c_n = \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\, dx. \tag{51}$$

The partial sums $S_nf, n \geq 1$, of a Fourier series of a function f are

$$S_nf(x) = \sum_{|k|\leq n} \hat{f}(n)e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x-y)\, f(y)\, dy, \tag{52}$$

where D_n is *Dirichlet kernel* of the form

$$D_n(x) = \sum_{|k|\leq n} e^{ikx} = \frac{\sin(n+1/2)x}{\sin(x/2)}, n = 0, 1, \ldots,$$

with value $2n+1$ at $x = 0$ (mod 2π). The Cesàro means $\sigma_n f$ are

$$\sigma_n f(x) = \frac{S_o f(x) + S_1 f(x) + \ldots + S_n f(x)}{n+1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x-y)\, f(y)\, dy,$$

where F_n denotes the *Fejér kernel*

$$F_n(x) = \frac{D_o f(x) + D_1 f(x) + \ldots + D_n f(x)}{n+1} = \frac{1}{n+1}\,\frac{\sin^2 \frac{n+1}{2}x}{\sin^2 \frac{x}{2}}$$

For $f \in L^p$ with $1 < p < \infty$ it yields that $\lim_{n\to\infty} \|f - S_nf\|_p = \lim_{n\to\infty} \|f - \sigma_n f\|_p = 0$ and for $f \in L^1$ we have $\lim_{n\to\infty} \|f - \sigma_n f\|_1 = 0$. On the other hand, if $f \in L^1$, then the Fourier sums can be divergent almost everywhere (Kolmogorov 1923) and the problem of almost everywhere convergence of Fourier series became an important object of research of many mathematicians around the world, including Marcinkiewicz and Zygmund.

4.4.1. *Pointwise convergence of Fourier series.* In this part we will investigate 2π-periodic functions, therefore functions or convergence of functions will be considered either on $[-\pi, \pi]$ or $[0, 2\pi]$.

The first paper of Marcinkiewicz [M33] contains a short proof of Kolmogorov's theorem (1924) on convergence of partial sums of lacunary Fourier series: if $f \in L^2$ and $\lambda_{n+1}/\lambda_n > q > 1$, then $S_{\lambda_n} f$ is convergent almost everywhere to f. A new proof deserved attention because of its brevity and clarity.

In the second paper, Marcinkiewicz [M34] generalized results of Wiener (1924) on the functions of finite p-variation with $p > 0$ (Wiener considered only the case $p = 2$). Recall that the function $f\colon [a, b] \to \mathbb{R}$ has finite *p-variation*, if

$$V_p(f) = V_p(f; a, b) = \left(\sup_{\Pi} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p\right)^{1/p} < \infty,$$

where supremum is taken over all partitions $\Pi\colon a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ of the interval $[a, b]$. The collection of all such functions of finite p-variation is denoted by $V_p[a, b]$, and by $V_p[0, 2\pi]$ we denote 2π-periodic functions having finite p-variation. Marcinkiewicz showed that ([M34], Thm 1 and 2) if $f \in V_p[a, b] (0 < p < \infty)$, then the function

$$\varphi_p(x) = \limsup_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|^{1/p}}$$

is finite almost everywhere and $\varphi_p \in L^p[a, b]$ with $\|\varphi_p\|_p \leq V_p[a, b]$. He also proved that if $f \in V_p[0, 2\pi](0 < p < \infty)$, then

$$\omega(\delta; f)_p := \sup_{|h| \le \delta} \left(\int_0^{2\pi} |f(x+h) - f(x)|^p \, dx \right)^{1/p} \le V_p(f; 0, 3\pi)\, \delta^{1/p} \text{ for } 0 < \delta \le \pi. \quad (53)$$

¿From this estimate we obtain the following results of Marcinkiewicz: *If $f \in V_p[0, 2\pi](p \ge 1)$, then the coefficients of the Fourier series of a function f are of order $O(n^{-1/p})$ and the Fourier series is convergent almost everywhere to f.*

The estimate (53) has appeared in textbooks and we can find them, for example, in [Ta79], pp. 17–18.

In the next paper [M35d] from 1935 Marcinkiewicz improved the result of Hardy–Littlewood (1932) on almost everywhere convergence of Fourier series.

THEOREM 21 (Marcinkiewicz test 1935). (a) *If, for x belonging to the set E of positive measure, we have*

$$\frac{1}{t} \int_0^t |f(x+u) - f(x)| \, du = O(\frac{1}{\log 1/|t|}), \quad (54)$$

then $(S_n f)$ is convergent almost everywhere in E.

(b) *If $f \in L^1[-\pi, \pi]$ and*

$$\int_0^{\pi} \omega_1(f, t) \frac{dt}{t} < \infty, \quad \textit{where} \quad \omega_1(f, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x+t) - f(x)| \, dx, \quad (55)$$

then $S_n f(x) \to f(x)$ for almost all $x \in [-\pi, \pi]$.

Marcinkiewicz also showed that the result in (a) is the best possible one, in the following sense (announced in [M35d] and proved in [M36a], Thm 2): if the function $\omega\colon \mathbb{R} \to (0, \infty)$ is even, nondecreasing in some interval $(0, \delta), 0 < \delta \le 1/3$ and such that $\lim_{t\to 0} \omega(t) = 0, \lim_{t\to 0} \omega(t) \log \frac{1}{|t|} = +\infty$, then there exists a function $f \in L^1$ for which

$$\frac{1}{t} \int_0^t |f(x+u) - f(x)| \, du = O[\omega(t)]$$

almost everywhere in $[0, 2\pi]$, but the Fourier series is divergent almost everywhere.

Theorem 21 (a) with proof can be found e.g. in the books by Zygmund ([Zy59], II, pp. 170–172), Alexis ([Al61], pp. 320–326) and Bary ([Ba64], I, pp. 417–421), where also is the proof of optimality of this theorem ([Ba64], I, pp. 443–447). Moreover, Theorem 21 (b) with proof can be found in the books written by Zygmund ([Zy59], II, p. 172), Torchinsky ([To86], p. 7), Bruckners and Thomson ([BBT97], p. 683).

Marcinkiewicz (1939) proved an interesting generalization of the Plessner theorem (1925), extending the case $p = 2$ to $1 \le p \le 2$ (for $p = 1$ we obtain the usual Dini test):

THEOREM 22 (Marcinkiewicz 1939). *Let $1 \le p \le 2$. If $f \in L^p$ and*

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|f(x+t) - f(x-t)|^p}{t} \, dt \, dx < \infty, \quad (56)$$

then the Fourier series of the function f is convergent almost everywhere.

Theorem 22, together with the proof can be found e.g. in the book by Bary [Ba64], I, pp. 379–380.

The year 1966 was a breakthrough for research on convergence almost everywhere of Fourier series. Then the Swedish mathematician Lennart Carleson [Ca66] proved the Luzin conjecture from 1913 that if $f \in L^2(-\pi,\pi)$, then the Fourier series is convergent almost everywhere. Equivalent formulation is: if $\{c_n\}_{n=-\infty}^{\infty} \in l^2$ is a sequence of complex numbers, then the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ is convergent almost everywhere. Hence, it can be deduced an analog for Fourier integrals: if $f \in L^2(\mathbb{R})$, then the functions $g_\alpha(x) = \int_{-\alpha}^{\alpha} f(t)e^{itx}\,dt$ are convergent almost everywhere, when $\alpha \to \infty$. In 1967, Richard Hunt, extended the result of Carleson to spaces $L^p(-\pi,\pi)$ for $1 < p < \infty$, and since then the assertion is called the Carleson–Hunt theorem. A new elegant proof of this theorem was given by Charles Fefferman [Fe73] in 1973.

I guess that the theorem which Marcinkiewicz might most have liked to see is the Carleson–Hunt theorem.

In the twenties appeared also papers on almost everywhere divergence of Fourier series. In 1923 Kolmogorov constructed a function $f \in L^1$ such that the partial Fourier sums $S_n f$ are unbounded almost everywhere, and so divergent almost everywhere. He also added an example of a function $f \in L^1$ which Fourier sums $S_{2^n} f$ are divergent almost everywhere. Three years later, Kolmogorov noticed the existence of a function $f \in L^1$ with divergent Fourier sums at every point.

In 1927 Kolmogorov together with Mienshov published the paper [KM27] in which they informed, without proof, about the existence of a function $f \in L^2$, whose Fourier series after a permutation of terms is divergent almost everywhere. It was not possible to reproduce their proof despite requests addressed even to Kolmogorov. Thus, there was not known neither a function nor permutation. A short method of construction was found only in 1960 by Z. Zahorski [Za60] (see also [Ul83], p. 71).

In 1936 Marcinkiewicz modified the construction of Kolmogorov from 1923 in his paper [M36a]. This necessary modification was not at all obvious.

Theorem 23 (Marcinkiewicz example 1936). *There exists a function $f \in L^1[0,2\pi]$ such that the Fourier sums $S_n f$ are divergent almost everywhere on $[0,2\pi]$ and*

$$\limsup_{n\to\infty} |S_n f(x)| < \infty$$

for almost all $x \in [0,2\pi]$, that is, the Fourier series of function f is boundedly divergent almost everywhere on $[0,2\pi]$.

The Kolmogorov and Marcinkiewicz constructions can be found in the Zygmund book [Zy59], pp. 305–308 and 308–310, in Bary book [Ba64], pp. 430–442, and in review articles of Ulyanov [Ul57], pp. 95–99 and 102–106, [Ul83], pp. 57–65.

[Al61] G. Alexits, *Convergence Problems of Orthogonal Series*, Pergamon Press, New York 1961 [4.7.11. Marcinkiewicz theorem, pp. 320–326].

[Ba64] N. K. Bary, *A Treatise on Trigonometric Series*, Vols. I, II, Pergamon Press, New York 1964.

[Ta79] R. Taberski, *Approximation of Functions by Trigonometric Polynomials*, Scientific Publ. Univ. of Adam Mickiewicz, Poznań 1979 (Polish).

[To86] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, Orlando 1986 [Theorem 1.4 (Marcinkiewicz), pp. 7–8].

[BBT97] A. M. Bruckner, J. B. Bruckner and B. S. Thomson, *Real Analysis*, Prentice-Hall, New Jersey 1997 [Theorem 15.20 (Marcinkiewicz), p. 683]

[Ca66] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. 116(1966), 135–157.

[Fe73] C. Fefferman, *Pointwise convergence of Fourier series*, Ann. of Math. (2) 98(1973), 551–571.

[Hu68] R. A. Hunt, *On the convergence of Fourier series*, in: "Orthogonal Expansions and their Continuous Analogues" (Proc. Conf., Edwardsville, Ill. 1967), Southern Illinois Univ. Press, Carbondale 1968, 235–255.

[Ko23] A. N. Kolmogoroff, *Une série de Fourier–Lebesgue divergente presque partout*, Fund. Math. 4(1923), 324–328.

[Ko26] A. N. Kolmogoroff, *Une série de Fourier–Lebesgue divergente partout*, C. R. Acad Sci. Paris 183(1926), 1327–1328.

[KM27] A. Kolmogoroff and D. Menchoff, *Sur la convergence des séries de fonctions orthogonales*, Math. Z. 26(1927), 432–441.

[La04] M. T. Lacey, *Carleson's theorem: proof, complements, variations*, Publ. Mat. 48(2004), no. 2, 251–307.

[Lu13] N. N. Luzin, *Sur la convergence des séries trigonométriques de Fourier*, C. R. Acad. Sci. Paris 156(1913), 1655–1658.

[Ul53] P. L. Ulyanov, *Generalization of a theorem of Marcinkiewicz*, Izvestiya Akad. Nauk SSSR. Ser. Math. 17(1953), 513–524.

[Ul57] P. L. Ulyanov, *On the divergence of Fourier series*, Uspekhi Mat. Nauk (N.S.) 12(1957), no. 3, 75–132 (Russian) [Marcinkiewicz example, pp. 102–106, Marcinkiewicz test, pp. 109–111 and its sharpness, pp. 111–116].

[Ul83] P. L. Ulyanov, *A. N. Kolmogorov and divergent Fourier series*, Uspekhi Mat. Nauk 38(1983), no. 4, 51–90 [Marcinkiewicz example, p. 65, Marcinkiewicz test, p. 68]; English transl. in Russian Math. Surveys 38(1983), no. 4, 57–100.

[Wi24] N. Wiener, *The quadratic variation of a function and its Fourier coefficients*, J. Math. and Phys. 3(1924), 72–94.

[Za60] Z. Zahorski, *Une série de Fourier permutée d'une fonction de classe L^2 divergente presque partout*, C. R. Acad. Sci. Paris 251(1960), 501–503.

[Zh87] H. S. Zhao, *The Marcinkiewicz theorem for Fourier series on compact Lie groups*, Chinese Ann. Math. Ser. A 8(1987), no. 6, 693–702; English summary in Chinese Ann. Math. Ser. B 9(1988), no. 1, 148–149.

4.4.2. *Orthogonal series.* A sequence $\Phi = (\varphi_n)_{n=1}^{\infty}$,where $\varphi_n \colon [a,b] \to \mathbb{R} (n = 1, 2, \ldots)$ is called an *orthogonal system* in $L^2[a,b]$, if

$$\int_a^b \varphi_m(t)\,\varphi_n(t)\,dt = 0 \quad \text{for } m \neq n \quad \text{and} \quad \int_a^b \varphi_n^2(t)\,dt = \lambda_n > 0 \ (m, n = 1, 2, \ldots).$$

If, in addition, $\lambda_1 = \lambda_2 = \ldots = 1$, then the system is called *orthonormal*.

We will discuss three results of Marcinkiewicz from the general theory of orthogonal series: the problem of almost everywhere convergence of a subsequence of any orthonormal system (1936), a generalization of Hausdorff–Young inequality (1937) and the theorem of Marcinkiewicz on Haar system (1937).

In 1936 Marcinkiewicz proved an interesting theorem for general orthonormal systems ([M36c], Thm A).

Theorem 24 (Marcinkiewicz 1936). *For any orthonormal system $\Phi = (\varphi_n)_{n=1}^{\infty}$ of functions defined on* $[0,1]$ *there exists an increasing sequence* $(N_k)_{k=1}^{\infty}$ *of natural numbers such that for any series*

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \quad \textit{with} \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty, \tag{57}$$

the sequence $\{S_{N_k}(x) = \sum_{n=1}^{N_k} a_n\varphi_n(x)\}_{k=1}^{\infty}$ *is almost everywhere convergent on* $[0,1]$ *and we have the estimate*

$$\| \sup_{m \geq 1} |S_{N_m}(x)| \|_2 \leq C(\sum_{n=1}^{\infty} |a_n|^2)^{1/2},$$

with constant C independent of the system Φ.

A subsequence $S_{N_k}(x)$ of partial sums of an orthogonal series, with N_k depending only on the system $\Phi = (\varphi_n)_{n=1}^{\infty}$. For different orthonormal systems the sequence (N_k) can be different. For example, for the Haar system we can take $N_k = k$ and for the trigonometric system $N_k = 2^k$.

If instead $(a_n) \in l^2$ we have a stronger assumption, then we can find a universal sequence (N_k) good for all orthonormal systems, for example, if $\sum_{n=1}^{\infty} |a_n|^2 \ln n < \infty$ and $f \in L^2[0,1]$ is a sum of series $\sum_{n=1}^{\infty} a_n\varphi_n(x)$ in the $\| \cdot \|_2$-norm, then the sequence $S_{2^n}(x) = \sum_{k=1}^{2^n} a_k\varphi_k(x)$ is convergent almost everywhere to $f(x)$.

By using Theorem 24 Marcinkiewicz showed the following ([M36c], Thm B): *if* $1 \leq p < \frac{6}{5}$, *then there is an* $f \in L^p[0,2\pi]$ *and a rearrangement of the Fourier series of f such that the new obtained series* $a_0/2 + \sum_{k=1}^{\infty}(a_{n_k} \cos n_k\, x + b_{n_k} \sin n_k\, x)$ *diverges unboundedly almost everywhere in* $[0,2\pi]$. In 1957 Ulyanov generalized this result to $1 \leq p < 2$ (see Bary [Ba64], I, pp. 480–482).

Marcinkiewicz Theorem 24 is cited e.g. in the following books:

[KS58] S. Kaczmarz and H. Steinhaus, *Theory of Orthogonal Series*, Fizmatgiz, Moscow 1958 (Russian).

[Si59] R. Sikorski, *Real Functions, Vol. II*, Monografie Matematyczne 37, PWN, Warszawa 1959 (Polish) [Marcinkiewicz result, p. 148].

Al61] G. Alexits, *Convergence Problems of Orthogonal Series*, Pergamon Press, New York 1961 [2.10.2. Marcinkiewicz–Menchoff theorem, pp. 152–155 and 158].

[Ga70] A. M. Garsia, *Topics in Almost Everywhere Convergence*, Lectures in Advanced Mathematics 4, Markham, Chicago 1970 [Marcinkiewicz theorem, pp. 79–80].

[KS84] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow 1984 (Russian) [Part 8, Theorem 7. Marcinkiewicz theorem with the proof and estimate $C \leq 6\sqrt{3}$, pp. 317–320].

Let $\Phi = (\varphi_n)_{n=1}^{\infty}$ be an arbitrary orthogonal system in $L^2(a,b)$ and let the interval (a,b) be finite or infinite. If $\|\varphi_n\|_2 \leq C$ for any $n \in \mathbb{N}$, then considering the Fourier coefficients

$$c_n = \int_a^b f(x)\, \varphi_n(x)\, dx \tag{58}$$

of the function f with respect to the system Φ we must assume that $f \in L^2(a,b)$ to ensure the existence of the integrals (58). If $\|\varphi_n\|_\infty \leq C$, then the integrals (58) exist for

$f \in L^1(a,b)$. Marcinkiewicz and Zygmund [MZ37] assumed that $\varphi_n \in L^r(a,b)$for$2 \le r \le \infty$. Then integrals (58) exist for $f \in L^{r'}$, where $\frac{1}{r} + \frac{1}{r'} = 1$.

Marcinkiewicz and Zygmund [MZ37b] gave the following generalization of the Hausdorff–Young theorem (for trigonometric system and $r = \infty$) and F. Riesz theorem (1923, for uniformly bounded orthonormal system and $r = \infty$).

THEOREM 25 (Marcinkiewicz and Zygmund 1937). *Let $\Phi = (\varphi_n)_{n=1}^{\infty}$ be an orthonormal system in $L^2(a,b)$ such that $\|\varphi_n\|_r = M_n < \infty$ for some $r \in (2,\infty]$ and arbitrary $n \in \mathbb{N}$. Assume also that p and q satisfy an equality $\frac{r'}{p} + \frac{2-r'}{q} = 1$.*

(a) *If $r' \le p \le 2$ and $f \in L^p(a,b)$, then $(\sum_{n=1}^{\infty} |c_n|^q \, M_n^{2-q})^{1/q} \le \|f\|_p$.*

(b) *If $2 \le p \le r$ and the sequence $a = (a_n)_{n=1}^{\infty}$ satisfy conditions*

$$I_q(a) = (\sum_{n=1}^{\infty} |a_n|^q \, M_n^{2-q})^{1/q} < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty,$$

then there exists $f \in L^p(a,b)$ such that $f = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \varphi_n$ in $L^p(a,b)$ and $\|f\|_p \le I_q(a)$.

Note that if the numbers M_n are bounded from below by a positive constant, then the inequality $\sum_{n=1}^{\infty} |a_n|^q \, M_n^{2-q} < \infty$ implies $\sum_{n=1}^{\infty} |a_n|^q < \infty$ and so $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ (assumption $2 \le p \le r$ and relation between p and q gives $1 \le q \le 2$). Hence, the condition of convergence $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ may be omitted in the statement of Theorem 25. This is, in particular, the case when the interval (a,b) is finite, because

$$M_n = \|\varphi_n\|_r \ge (b-a)^{1/r-1/2} \|\varphi_n\|_2 = (b-a)^{1/r-1/2}.$$

Marcinkiewicz and Zygmund also proved a generalization of the Paley theorem (1931, $r = \infty$ and $M_1 = M_2 = \ldots = M$).

THEOREM 26 (Marcinkiewicz and Zygmund 1937). *Let $\Phi = (\varphi_n)_{n=1}^{\infty}$ be an orthonormal system in $L^2(a,b)$ such that $\|\varphi_n\|_r = M_n < \infty$ for some $r \in (2,\infty]$ and arbitrary $n \in \mathbb{N}$. Assume also that $M_1 \le M_2 \le \ldots M_n \le \ldots$.*

(a) *If $r' < p \le 2$ and $f \in L^p(a,b)$, then $(\sum_{n=1}^{\infty} |c_n|^p \, M_n^{\frac{r}{r-2}(p-2)} \, n^{\frac{r-1}{r-2}(p-2)})^{1/p} \le A(p,r)\, \|f\|_p$.*

(b) *If $2 \le p < r$ and the sequence $a = (a_n)_{n=1}^{\infty}$ satisfies the condition*

$$J_{r,p}(a) = (\sum_{n=1}^{\infty} |a_n|^p \, M_n^{\frac{r}{r-2}(p-2)} \, n^{\frac{r-1}{r-2}(p-2)})^{1/p} < \infty,$$

then there exists $f \in L^p(a,b)$ such that $f = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \varphi_n$ in $L^p(a,b)$ and $\|f\|_p \le B(p,r) J_{r,p}(a)$.

(c) *Moreover, $B(p,r) \le C\, \frac{r-2}{r-p}\, p$ for some $C > 0$ and $A(p,r) = B(p',r)$.*

Marcinkiewicz and Zygmund obtained the proof of Theorem 25 by using the interpolation theorem of M. Riesz. In [Zy56] Zygmund proved the original Paley theorem using the Marcinkiewicz interpolation theorem between strong type $(2,2)$ and weak type $(1,1)$ of the linear operator $Tf = (nc_n) = (n \int_a^b f(t)\, \varphi_n(t)\, dt)$. An application of the Marcinkiewicz interpolation theorem give also some other generalizations of the Hausdorff–Young theorem (see Edwards [Ed82], pp. 192–197).

In [SW58] Stein and Weiss first proved a theorem on interpolation of operators with change of measures, as a generalization of Riesz–Thorin and Marcinkiewicz theorems, and then used it to prove weighted version of the Paley theorem with $M_n^* = \max_{1\le k\le n} M_k$ instead of M_n with the increases monotonically assumption. Bullen [Bu61] considers, firstly, the cases of equality in the Hausdorff–Young theorems and then also observed that the assumption of increasing monotonicity of $\{M_n\}$ can be a little weaker, namely that for some $a > 1$ and all $m, n \in \mathbb{N}, m < n$ we have $\max_{a^m+1\le k\le a^{m+1}} M_k \le C \max_{a^n+1\le k\le a^{n+1}} M_k$. Moreover, Kirillov [Ki98] proved that the monotone increasing assumption on the sequence (M_n) cannot be discarded.

In [Ko92] Kolyada gave a proof of the Marcinkiewicz–Zygmund Theorem 26(b) using elementary inequalities for numbers and also extended this theorem to Orlicz classes. Theorem 26 with new estimates in Lorentz spaces $L^{p,q}$ were given by Kirillov [Ki98].

The Hausdorff–Young and Riesz theorems were also considered for some concrete nonorthogonal systems but similar to orthogonal ones, as, for example, S. Verblunsky (1954) considered in $L^2[0, 2\pi]$ the system $(\exp(i\chi_n x))_{n\in\mathbb{Z}}$, where $\chi_{-n} = -\chi_n$ and $0 = \chi_0 < \chi_1 < \ldots$ are positive solutions of the equation $x + h\tan \pi x = 0$ with $h > 0$ fixed. In [Ro01] Rodionov studied expansions of functions in the space L^p with respect to systems similar to orthogonal ones to include also the result by Verblunsky and proved Marcinkiewicz–Zygmund type theorems for such systems.

[Ed82] R. E. Edwards, *Fourier Series. Vol. 2. A Modern Introduction*, 2nd ed., Springer, New York-Berlin 1982.

[Bu61] P. S. Bullen, *Properties of the coefficients of orthonormal sequences*, Canad. J. Math. 13(1961), 305–315.

[Ki98] S. A. Kirillov, *On a theorem of Marcinkiewicz and Zygmund*, Mat. Zametki 63(1998), no. 3, 386–390; English transl. in Math. Notes 63(1998), no. 3–4, 338–341.

[Ki98] S. A. Kirillov, *Norm estimates of functions in Lorentz spaces*, Acta Sci. Math. (Szeged) 65(1999), no. 1–2, 189–201.

[Ko92] V. I. Kolyada, *Some generalizations of the Hardy–Littlewood–Paley theorem*, Mat. Zametki 51(1992), no. 3, 24–34; English transl. in Math. Notes 51(1992), no. 3–4, 235–244.

[Pa31] R. E. A. C. Paley, *Some theorems on orthogonal functions*, Studia Math. 3(1931), 226–238.

[Ri23] F. Riesz, *Über eine Verallgemeinerung der Parsevalschen Formel*, Math. Z. 18(1923), 117–124.

[Ro01] T. V. Rodionov, *Analogues of the Hausdorff–Young and Hardy–Littlewood theorems*, Izv. Ross. Akad. Nauk Ser. Mat. 65(2001), no. 3, 175–192; English transl. in Izv. Math. 65(2001), no. 3, 589–606.

[SW58] E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. 87(1958), 159–172.

One of the classical orthonormal systems of functions is the *Haar system* $\{h_n\}_{n=1}^{\infty}$ in which the functions are defined on the unit interval $[0, 1]$ in the following way: $h_1(x) = 1$ for all $x \in [0, 1]$ and if $n = 2^m + k, k = 1, 2, \ldots, 2^m, m = 0, 1, \ldots$

$$h_n(x) = \begin{cases} 2^{m/2}, & \text{if } x \in (\frac{2k-2}{2^{m+1}}, \frac{2k-1}{2^{m+1}}), \\ -2^{m/2}, & \text{if } x \in (\frac{2k-1}{2^{m+1}}, \frac{2k}{2^{m+1}}), \\ 0, & \text{if } x \notin (\frac{k-1}{2^m}, \frac{k}{2^m}). \end{cases}$$

At interior points of discontinuity a Haar function is put equal to half the sum of its limiting values from the right and from the left, and at the end points of $[0,1]$ to its limiting values from within the interval.

Schauder [Sc28] proved that Haar system is a basis (Schauder basis) in $L^p[0,1]$ for $1 \le p < \infty$. Marcinkiewicz continued studies of the Haar system by showing in [M37a] that it is an unconditional basis for $1 < p < \infty$.

Theorem 27 (Marcinkiewicz 1937). *For $1 < p < \infty$ the Haar system is an unconditional basis in $L^p[0,1]$, that is, it remains a basis under any permutation of its elements.*

This theorem is sometimes called the Paley–Marcinkiewicz theorem since Marcinkiewicz proof is a consequence of Paley's results on the Walsh system. Gundy [Gu67], Burkholder [Bu73] and Gapoškin [Ga74] gave simple proofs of this theorem. A generalization of Marcinkiewicz Theorem 27 on reflexive Orlicz spaces was presented by Gapoškin [Ga67], [Ga68], and on separable symmetric spaces X on $[0,1]$ with Boyd indices $0 < \alpha_X \le \beta_X < 1$ by Semenov [Se69] (this theorem with the proof we can find in the books [KPS82], pp. 181–182 and [LT79], pp. 156–158).

More information and the proof of Theorem 27 can be found in the books and papers cited below:

[KS58] S. Kaczmarz and H. Steinhaus, *Theory of Orthogonal Series*, Fizmatgiz, Moscow 1958 (Russian) [Marcinkiewicz theorems, pp. 449–450].

[Si70] I. Singer, *Bases in Banach Spaces I*, Springer, Berlin 1970 [Theorem 14.1. Marcinkiewicz theorem, pp. 407–409 and 633].

[Ol75] A. M. Olevskii, *Fourier Series with Respect to General Orthogonal Systems*, Springer, New York 1975 [Marcinkiewicz, p. 71].

[LT79] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. II. Function Spaces*, Springer, Berlin 1979 [Theorem 2.c.5, pp. 155–156].

[KPS82] S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, AMS, Providence 1982 [Theorem 9.6. for the L_p spaces was obtained by Marcinkiewicz, pp. 181–182 and 351].

[KS84] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow 1984 (Russian) [Part 3, Theorem 8. Marcinkiewicz theorem, pp. 92].

[NS97] I. Novikov and E. Semenov, *Haar Series and Linear Operators*, Kluver Acad. Publ., Dordrecht 1997 [Marcinkiewicz theorem, pp. 36–39].

[Bu73] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probab. 1(1973), 19–42.

[Ga67] V. F. Gapoškin, *The existence of unconditional bases in Orlicz spaces*, Funkcional. Anal. i Prilozhen 1(1967), no. 4, 26–32; English transl. in Functional Anal. Appl. 1(1967), no. 4, 278–284.

[Ga68] V. F. Gapoškin, *Unconditional bases in Orlicz spaces*, Sibirsk. Mat. Zh. 9(1968), no. 2, 280–287; English transl. in Siberian Math. J. 9(1968), no. 2, 211–217.

[Ga74] V. F. Gapoškin, *The Haar system as an unconditional basis in $L_p[0,1]$*, Mat. Zametki 15(1974), no. 2, 191–196; English transl. in Math. Notes 15(1974), no. 2, 108–111.

[Go70] B. I. Golubov, *Series in the Haar system*, Itogi Nauki i Tekhniki, Mat. Analiz 1970, Moscow 1971, 109–146; English transl. in J. Soviet Math. (New York) 1(1973), no. 6, 704–726.

[Gu67] R. F. Gundy, *The martingale version of a theorem of Marcinkiewicz and Zygmund*, Ann. Math. Statist 38(1967), 725–734.

[GU58] R. S. Guter and P. L. Ulyanov, *On some results in the theory of orthogonal series*, supplement to the translation from German into Russian of the book "S. Kaczmarz and H. Steinhaus, Theorie der Orthogonalreihen, Warszawa-Lwów 1935", GIFML, Moscow 1958 (Russian) [Marcinkiewicz theorem on Haar functions, pp. 449–450]; English transl. *Supplement to theory of orthogonal series*, Amer. Math. Soc. Transl., II. Ser. 17 (1961), 219–250.

[Ha10] A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Math. Ann. 69(1910), 331–371.

[Kr78] V. G. Krotov, *Unconditional convergence of the Fourier series in the Haar system in the spaces* Λ^p_ω, Mat. Zametki 23(1978), no. 5, 685–695; English transl. in Math. Notes 23(1978), no. 5–6, 376–382.

[Pa32] R. E. A. C. Paley, *A remarkable system of orthogonal functions*, Proc. Lond. Math. Soc. 34(1932), No. 2, 241–279.

[Pe85] A. Pełczyński, *Norms of classical operators in function spaces*, Astérisque 131(1985), 137–162 [2. The Marcinkiewicz–Paley inequality for the Haar system, pp. 144–146].

[Sc28] J. Schauder, *Eine Eigenschaft des Haarschen Orthogonalsystems*, Math. Z. 28(1928), no. 1, 317–320.

[Se69] E. M. Semenov, *A certain method for obtaining interpolation theorems in symmetric spaces*, Dokl. Akad. Nauk SSSR 185(1969), 1243–1246; English transl. in Soviet Math. Dokl. 10(1969), 507–511.

[Ul61] P. L. Ulyanov, *Divergent Fourier series*, Uspekhi Mat. Nauk 16(1961), no. 3, 61–142 [Theorem D. Marcinkiewicz result, pp. 68–69]; English transl. in Russian Math. Surveys 16(1963), no. 3, 1–74.

[Wa91] G. Wang, *Sharp square-function inequalities for conditionally symmetric martingales*, Trans. Amer. Math. Soc. 328(1991), no. 1, 393–419.

4.5. Approximation theory. Approximation of functions by Fourier series we discussed earlier. Now, we instead concentrate on Lagrange type interpolation and trigonometric interpolation.

4.5.1. *Marcinkiewicz theorem on Lagrange interpolation.* The Lagrange interpolation problem is to construct, for a given continuous function $f\colon [a,b] \to \mathbb{R}$ and different n points (nodes) $x_1, x_2, \dots, x_n$ from the interval $[a,b]$, a polynomial $L_{n-1}(f,x)$ of degree at most $n-1$ such that $L_{n-1}(f,x_k) = f(x_k)$ for $k = 1,2,\dots,n$. A polynomial of this type can be found by using Lagrange's interpolation formula. Then also an estimate on the error $f(x) - L_{n-1}(f,x)$ is of interest.

Interpolation process is given on $[a,b]$ if it is given infinite triangular matrix of nodes (table of nodes) $(x_{n,k})_{1\le k\le n, 1\le n}$ such that in every row we have different points $x_{n,k} \neq x_{n,j}$ for $k \neq j$ from interval $[a,b]$.

We do not necessarily have $L_{n-1}(f,x) \to f(x)$, but S. Bernstein constructed (for certain interpolation points) other interpolation polynomials $A_n(f,x)$ of degree $m_n > n$ such that $A_n(f,x) \to f(x)$. G. Faber (1914) has shown that there is no table of nodes that the corresponding interpolation process is uniformly convergent for any continuous function, since for any table of nodes on $[-1,1]$ there is $f \in C[-1,1]$ for which a sequence of Lagrange polynomials $L_{n-1}(f,x)$ satisfies $\limsup_{n\to\infty} |L_{n-1}(f,x)| = \infty$. On the other hand, we have Marcinkiewicz theorem concerning the possibility to construct a table of nodes for every function separately (cf. [M36e], Thm 3).

THEOREM 28 (Marcinkiewicz 1936). *For any continuous function f on $[a,b]$ there exists an infinite triangular matrix of nodes such that the corresponding interpolation process for f is uniformly convergent.*

Theorem 28 can be found, for example, in the book by Daugavet [Da77], pp. 153–154.

4.5.2. *Grünwald–Marcinkiewicz interpolation theorem.* Let be given $2n+1$ distinct points $x_0, x_1, \ldots, x_{2n}$ on the x-axis, mod 2π, and any $2n+1$ real numbers $y_0, y_1, \ldots, y_{2n}$. Then there exists a trigonometric polynomial

$$T_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

of degree n (it has $2n+1$ coefficients) such that $T_n(x_k) = y_k (k = 0, 1, \ldots, 2n)$ and such a polynomial is unique. The most interesting case is when the points $e^{ix_0}, e^{ix_1}, \ldots, e^{ix_{2n}}$ are equally distributed on the unit circle, that is, when $x_k = x_0 + \frac{2\pi k}{2n+1}$, and the numbers $y_0, y_1, \ldots, y_{2n}$ are the values of a continuous function $f(x)$ at the points $x_0, x_1, \ldots, x_{2n}$. The polynomial $T_n(x) = T_n(f, x)$ is then called the *Lagrange interpolating polynomial* for the function f; it depends also on the point x_0, but for the sake of simplicity we may fix the point x_0 once for all and take $x_0 = 0$. Polynomials T_n are given by the formulas

$$T_n f(x) = \frac{1}{\pi}\int_0^{2\pi} f(t)\frac{\sin(n+\frac{1}{2})(x-t)}{2\sin\frac{1}{2}(x-t)}\, d\omega_{2n+1}(t), \tag{59}$$

where $\omega_{2n+1}(t)$ is a step function having jumps $2\pi/(2n+1)$ at the points x_k and is continuous elsewhere. If we replace here $d\omega_{2n+1}(t)$ by dt we obtain the classical formula for the nth partial sums $S_n f(x)$ of the Fourier series of the function f. Since when $n \to \infty$ the graph of $\omega_{2n+1}(t)$ approaches, after the substraction of a suitable constant, the limit $g(t) = t$, it is natural to conjecture that the behaviour of the sequence (T_n) is for $n \to \infty$ similar to the behaviour of the sequence (S_n).

It is actually so within certain limits. For example, already Faber showed that there exists a continuous function f such that the sequence $\{S_n f(x)\}$ is uniformly convergent while the sequence $\{T_n(f, x)\}$ diverges at certain individual points. In 1933, Marcinkiewicz in his PhD [M35b] constructed *a continuous function f such that the sequence $\{S_n f(x)\}$ converges uniformly while the sequence $\{T_n(f, x)\}$ diverges almost everywhere.*

At the time when the dissertation was published (as a paper [M35c]), it appeared the paper [Gr35] of the Hungarian mathematician Géza Grünwald[13] and it is containing a similar result for the so-called Tchebyshev interpolation, which differs only formally from the Lagrange interpolation. Matrix of nodes in Tchebyshev interpolation is given by points $\{x_{n,k} = \cos\frac{2k-1}{2n}\pi\}_{1\le k\le n, 1\le n}$ on the interval $[-1, 1]$, and an interpolation polynomial by $L_{n-1}(f, x)$.

It is curious that a year later both authors could, independently of each other, strengthen their examples by constructing continuous functions whose Tchebyshev interpolating polynomials diverge everywhere.

[13] Géza Grünwald (born 18 October 1910 in Budapest – killed 7 September 1942, as holocaust victim).

THEOREM 29 (Grünwald 1936, Marcinkiewicz 1936/37). *There exists a function $f \in C[-1,1]$ such that the sequence $\{L_{n-1}(f,x)\}_{n=1}^{\infty}$ is divergent at every point $x \in [-1,1]$, i.e.,* $\limsup_{n\to\infty} |L_{n-1}(f,x)| = \infty$ *for any $x \in [-1,1]$.*

The proof of Theorem 29 is given, for example, in the books of Zygmund ([Zy59], Vol. II, pp. 44–46) and Natanson [Na65]. The topic was developed later by G. Grünwald, A. A. Privalov, P. Turán, P. Erdös and P. Vértesi. In particular, the last two authors have shown in [EV80] the Grünwald–Marcinkiewicz theorem for any arrangement of nodes.

Theorem 29 is important for two reasons. On the one hand it shows that the Lagrange approximation method is sometimes not a good approximation, even at the nodes. On the other hand, we can see many similarities between approximation by the Fourier sums $S_n f$ and the Lagrange polynomial interpolation $L_{n-1}(f,\cdot)$. These similarities could be used, although to a limited extent. For example, Marcinkiewicz in his master thesis noticed that a *continuous function f for which the sequence $\{T_n f(x)\}$ diverges almost everywhere (or even everywhere) can satisfy the condition $f(x+h) - f(x) = O(1/\log(1/h))$, i.e., the sequence $\{S_n f(x)\}$ is convergent almost everywhere.*

Replacing O by o the sequence $\{T_n f(x)\}$ converges uniformly. Another example, is the Carleson theorem (1964), which shows that if $f \in C[-1,1]$, then the partial Fourier sums $S_n f(x)$ converges to $f(x)$ almost everywhere on $[-1,1]$, but from Theorem 28 this is not the case for the sequence $\{T_n f(x)\}$. Marcinkiewicz proved in [M38i] that if $f \in L^1$ is periodic and F is the indefinite integral of the function f, then the derivatives of $T_n'(F,x)$ converges to $f(x)$ and better imitate the behavior of the partial sums $S_n f(x)$ than the polynomials $T_n(f,x)$.

It is also worth to mention about the following result of Marcinkiewicz from 1936 ([M36b], Thm 1): *there is a continuous 2π-periodic function f such that the arithmetic means*

$$\frac{T_0 f(x) + T_1 f(x) + \ldots + T_n f(x)}{n+1}$$

diverge at some points. This means that we do not have an obvious analogue of the classical theorem of Fejér about the arithmetic means of the partial sums of Fourier series.

This theorem, proofs and generalizations can be found in the following books and papers:

[Na55] I. P. Natanson, *Konstruktive Funktionentheorie*, Akademie, Berlin 1955 [II.3. Ein Beispiel von Marcinkiewicz, pp. 379–388].

[Ch66] E. W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York 1966 [information on Marcinkiewicz [1937] and Grünwald [1936] result, p. 233].

[Da75] P. J. Davis, *Interpolation and Approximation*, Dover, New York 1975 [Marcinkiewicz, p. 79].

[Da77] I. K. Daugavet, *Introduction to the Theory of Approximation of Functions*, Izdat. Leningrad. Univ., Leningrad 1977 (Russian) [Chapter 5, Theorem 1 (Marcinkiewicz), pp. 153–154].

[SV90] J. Szabados and P. Vértesi, *Interpolation of Functions*, World Scientific, Singapore 1990 [Grünwald–Marcinkiewicz result, p. 126].

[Ti94] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Dover, New York 1994 [information on Marcinkiewicz and Grünwald result, p. 579].

[EV80] P. Erdös and P. Vértesi, *On the almost everywhere divergence of Lagrange interpolatory polynomials for arbitrary system of nodes*, Acta Math. Acad. Sci. Hungar. 36(1980), no. 1–2, 71–89.

[Gr35] G. Grünwald, *Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome*, Acta Sci. Math. (Szeged) 7(1935), 207–221.

[Gr36] G. Grünwald, *Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome Stetiger Funktionen*, Ann. Math. 37(1936), 908–918.

[Gr43] G. Grünwald, *On the theory of interpolation*, Acta Math. 75(1943), 219–245.

[MV01] T. M. Mills and P. Vértesi, *An extension of the Grünwald–Marcinkiewicz interpolation theorem*, Bull. Austral. Math. Soc. 63(2001), no. 2, 299–320.

[Of40] A. C. Offord, *Approximation to functions by trigonometric polynomials*, Duke Math. J. 6(1940), no. 2, 505–510.

[Re03] M. Revers, *A survey on Lagrange interpolation based on equally spaced nodes*, in: Advanced Problems in Constructive Approximation, Birkhäuser, Basel 2003, 153–163.

4.5.3. *Marcinkiewicz–Zygmund inequalities.* In the proofs on the convergence of Lagrange interpolation, there are some inequalities used in the estimation error of Lagrange interpolation – they compare a continuous norm and its discretization. We are talking about the results originating in papers of Marcinkiewicz [M36b] and Marcinkiewicz–Zygmund [MZ37a].

THEOREM 30 (Marcinkiewicz inequalities 1936). *If* $1 < p < \infty$, *then there exist constants* $A_p, B_p > 0$ *such that for arbitrary trigonometric polynomial* T *of degree* $\leq n$ *the following inequalities hold:*

$$\frac{A_p}{2n+1}\sum_{k=0}^{2n}|T(\frac{2k\pi}{2n+1})|^p \leq \int_0^{2\pi}|T(x)|^p\,dx \leq \frac{B_p}{2n+1}\sum_{k=0}^{2n}|T(\frac{2k\pi}{2n+1})|^p.$$

THEOREM 31 (Marcinkiewicz–Zygmund inequalities 1937). *If* $1 < p < \infty$, *then there exist constants* $C_p, D_p > 0$ *such that for any complex polynomial* P *of degree* $\leq n$ *we have the following inequalities*

$$\frac{C_p}{n+1}\sum_{k=0}^{n}|P(e^{i\frac{2\pi k}{n+1}})|^p \leq \int_0^{2\pi}|P(e^{i\theta})|^p\frac{d\theta}{2\pi} \leq \frac{D_p}{n+1}\sum_{k=0}^{n}|P(e^{i\frac{2\pi k}{n+1}})|^p.$$

Note that for $p = 1$ and $p = \infty$ the first inequalities in Theorems 30 and 31 are still true, but the second ones are not true. Proofs of the inequalities in Theorem 30 can be found in Marcinkiewicz's paper ([Ma36b], Thm 9 and 10) and Marcinkiewicz–Zygmund paper ([MZ37a], Thm 1 and 2), and also in the book by Zygmund ([Zy59], Thm 7.5). Proofs of Theorem 31 are in the paper by Marcinkiewicz and Zygmund ([MZ37a], Thm 10) and the Zygmund's book ([Zy59], Thm 7.10).

The above inequalities have been generalized in different ways, see the papers cited below.

[Mi70] D. S. Mitrinović, *Analytic Inequalities*, Springer, Berlin 1970 [3.5.35. Marcinkiewicz and Zygmund inequalities, p. 261].

[Ku04] J. C. Kuang, *Applied Inequalities*, 3rd ed., Shangdong 2004 (Chinese) [42. Marcinkiewicz–Zygmund inequalities, p. 432].

[BKP09] A. Böttcher, S. Kunis and D. Potts, *Probabilistic spherical Marcinkiewicz–Zygmund inequalities*, J. Approx. Theory 157(2009), no. 2, 113–126.

[CZ99] C.K. Chui, L. Zhong, *Polynomial interpolation and Marcinkiewicz–Zygmund inequalities on the unit circle*, J. Math. Anal. Appl. 233(1999), no. 1, 387–405.

[Da03] S. B. Damelin, *Marcinkiewicz–Zygmund inequalities and the numerical approximation of singular integrals for exponential weights: methods, results and open problems, some new, some old*, J. Complexity 19(2003), no. 3, 406–415.

[DJK02] S. B. Damelin, H. S. Jung and K. H. Kwon, *Converse Marcinkiewicz–Zygmund inequalities on the real line with application to mean convergence of Lagrange interpolation*, Analysis (Munich) 22(2002), no. 1, 33–55.

[Lu97] D. S. Lubinsky, *Marcinkiewicz–Zygmund inequalities: methods and results*, in: "Recent Progress in Inequalities" (ed. G. V. Milovanović), Kluwer, Dordrecht 1997, 213–240.

[Lu99] D. S. Lubinsky, *On Converse Marcinkiewicz–Zygmund inequalities in* $L_p, p > 1$, Const. Approx. 15(1999), 577–610.

[Ma07] J. Marzo, *Marcinkiewicz–Zygmund inequalities and interpolation by spherical harmonics*, J. Funct. Anal. 250(2007), no. 2, 559–587.

[MR00] G. Mastroianni and M. G. Russo, *Weighted Marcinkiewicz inequalities and boundedness of the Lagrange operator*, in: "Mathematical Analysis and Applications" (ed. T. M. Rassias), Hadronic Press, Palm Harbor 2000, 149–182.

[MN00] H. N. Mhaskar, F. J. Narcowich and J.D. Ward, *Spherical MarcinkiewiczZygmund inequalities and positive quadrature*, Math. Comput. 70(2000) 1113–1130.

[MP98] H. N. Mhaskar and J. Prestin, *On Marcinkiewicz–Zygmund-type inequalities*, in: "Approximation Theory" (in memory of A. K. Varma), Marcel Dekker, New York 1998, 389–404.

[OS07] J. Ortega-Cerdà and J. Saludes, *Marcinkiewicz–Zygmund inequalities*, J. Approx. Theory 145(2007), no. 2, 237–252.

[RS98] K. V. Runovskii and H. J. Schmeisser, *On Marcinkiewicz–Zygmund type inequalities for irregular knots in* L_p*-spaces,* $0 < p \leq +\infty$, Math. Nachr. 189(1998), 209–220.

[Sc08] D. Schmid, *Marcinkiewicz–Zygmund inequalities and polynomial approximation from scattered data on* SO(3), Numer. Funct. Anal. Optim. 29(2008), no. 7–8, 855–882.

[Xu91] Y. Xu, *The generalized Marcinkiewicz–Zygmund inequality for trigonometric polynomials*, J. Math. Anal. Appl. 161(1991), no. 2, 447–456.

[Xu91c] Y. Xu, *On the Marcinkiewicz–Zygmund inequality*, in: "Progress in Approximation Theory" (eds. P. Nevai and A. Pinkus), Academic Press, Boston 1991, 879–891.

[ZS94] L. Zhong and X. Shen, *Weighted Marcinkiewicz–Zygmund inequalities*, Adv. in Math. (China) 23(1994), 66–75.

4.6. Some other Marcinkiewicz results. Here are chosen six other themes in which there are results of Marcinkiewicz, and which were cited in some books and papers.

4.6.1. *Strong summability of Fourier series.* From the Fejér theorem, for every $f \in L^1$ we have convergence almost everywhere of $\sigma_n f(x) \to f(x)$, when $n \to \infty$, i.e.

$$\frac{1}{n+1} \sum_{m=0}^{n} [S_m f(x) - f(x)] = o(1) \quad \text{a. e.}$$

Hardy and Littlewood (1927) asked for the truth of the following stronger property

$$\frac{1}{n+1}\sum_{m=0}^{n}|S_m f(x)-f(x)|=o(1) \quad \text{a. e.},$$

and then for strong summability H_r in the sense of Hardy of order $r>0$, i.e., the property

$$\frac{1}{n+1}\sum_{m=0}^{n}|S_m f(x)-f(x)|^r=o(1) \quad \text{a. e.} \tag{60}$$

By the Hölder–Rogers inequality we can easily see that for larger r the result is stronger.

In 1939 Marcinkiewicz [M39d] gave a positive answer to the problem of Hardy–Littlewood in the case $r=2$, proving that: *if $f\in L^1$, then*

$$\frac{1}{n+1}\sum_{m=0}^{n}[S_m f(x)-f(x)]^2=o(1) \quad \textit{a. e.}$$

This theorem with a proof is e.g. in the book of Bary ([Ba64], II, pp. 24–31). Zygmund [Zy42] generalized this theorem to any $r>0$ in the property (60) and his proof is completely different (cf. also Zygmund book [Zy59], II, pp. 185–186).

[Zy42] A. Zygmund, *On the convergence and summability of power series on the circle of convergence. II*, Proc. London Math. Soc. (2) 47(1942), 326–350.

[Ta55] K. Tandori, *On strong summability of Fourier series*, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5(1955), 457–465 (Hungarian).

4.6.2. *Absolutely convergent Fourier series.* A theorem of P. Lévy, generalizing a theorem of N. Wiener (see Zygmund [Zy59], I, p. 245 and Bary [Ba64], II, pp. 190–194), states that if $g(x)$ has an absolutely convergent Fourier series, and if $f(x)$ is analytic in the closed interval $(\min g(x), \max g(x))$, then $f[g(x)]$ also has an absolutely convergent Fourier series. Marcinkiewicz showed in [M40] that this result can be extended by requiring less on $f(x)$ and more on $g(x)$.

For $0<s\le 1$ and an open interval $I\subset\mathbb{R}$ denote by $G_s(I)$ the class of infinitely differentiable functions F on I satisfying inequalities of the type $|F^{(n)}(x)|\le B^n n^{n/s}$ on every closed subinterval of I. For $0<p<\infty$ denote by A_p the class of functions $f(t)=\sum_{n\in\mathbb{Z}}a_n e^{int}$ such that $\sum_{n\in\mathbb{Z}}|a_n|^p<\infty$. Marcinkiewicz proved that: *if $f(t)\in A_s$ $(0<s\le 1)$, I contains the range of f and $F\in G_s(I)$, then $F(f)\in A_1$*. Zygmund has pointed out that the proof of Marcinkiewicz can be extended to show that in fact $F(f)\in A_s$. This result is called the *Wiener–Lévy–Marcinkiewicz theorem* on absolutely convergent series. Rivière and Sagher [RS66] proved the converse of Marcinkiewicz's theorem, in stronger form. Marcinkiewicz method was used by Kahane, Katznelson, Mallivan and others in the so-called "symbolic calculus" in or between the algebras A_p. More information can be found in [Ka68] and [Ka70].

The interest in the space A_p for $0<p<1$ follows from the fact that this is a nontrivial example of locally bounded algebra which is not a Banach algebra.

Marcinkiewicz also showed that ([M40]; see also Bary [Ba64], II, pp. 194–196): *there exist functions $f(x)$ and $g(x)$, both having absolutely convergent Fourier series, but such that the Fourier series of $f[g(x)]$ does not converge absolutely. The function $f(x)$ is* *equal to zero in* $(-\pi, 0)$ *and at* π*, equal to* $(\log\ x)^{-2}$ *in* $(0, \frac{1}{2})$*, and linear in* $(\frac{1}{2}, \pi)$*; and* $g(x) = f(x)$.

[Ka70] J.-P. Kahane, *Séries de Fourier Absolument Convergentes*, Springer, Berlin-Heidelberg-New York 1970 [VI.4. Method of Marcinkiewicz, pp. 77–80]; Russian transl. in Mir, Moscow 1976 [VI. 4. Method of Marcinkiewicz, pp. 96–99].

[Dy71] E. M. Dynkin, *Individual theorems of Wiener–Levy type for Fourier series and integrals*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 22(1971), 181–182 (Russian).

[Ka68] J.-P. Kahane, *Sur les séries de Fourier à coefficients dans* l^p, in "Orthogonal Expansions and their Continuous Analogues" (Proc. Conf., Edwardsville, Ill., 1967), Southern Illinois Univ. Press, Carbondale, Ill. 1968, 257–272.

[RS66] N. M. Rivière and Y. Sagher, *The converse of Wiener–Levy–Marcinkiewicz theorem*, Studia Math. 28(1966), 133–138.

[Ul02] P. L. Ulyanov, *On Lévy and Marcinkiewicz theorems for Fourier–Haar series*, Izv. Nats. Akad. Nauk Armenii Mat. 36(2001), no. 4, 73–81 (2002); English transl. in J. Contemp. Math. Anal. 36(2001), no. 4, 77–85 (2002).

4.6.3. *Thin sets related to trigonometric series.* A set $E \subset [0, 2\pi]$ is called a *set of uniqueness*, or a *U-set*, if any trigonometric series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ which converges to zero for $x \notin E$ is identically zero; that is, such that $a_n = b_n = 0$ for all n. Otherwise E is a *set of multiplicity* (sometimes called an *M-set* or a *Menshov set*). If $E \subset [0, 2\pi]$ is an M-set, then there is a trigonometric series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ that converges to zero on $[0, 2\pi]\backslash E$ and that has nonzero coefficients. Analogous definitions apply on the real line, and in higher dimensions. Every countable set is a U-set. Every set E of positive measure is an M-set

Marcinkiewicz and Zygmund proved that ([MZ37d]; see also Bary [Ba64], II, p. 364): *If* E *is a U-set and* $\theta > 0$ *is such that* $E(\theta) = \{\theta x : x \in E\} \subset [0, 2\pi]$*, then* $E(\theta)$ *is a U-set.* For example, for $0 < \theta < 1$ the set $E(\theta)$ is again a U-set.

Marcinkiewicz [M38f], in honour of V. V. Nemytzkii, introduced the notion of an N-set (investigated earlier by P. Fatou (1906) and A. Rajchman (1922)): a set $A \subset [0, 1]$ is an *N-set* if there is a trigonometric series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx)$ which converges absolutely in A but not everywhere, that is, $\sum_{n=1}^{\infty}(|a_n| + |b_n|) = \infty$. Equivalently (see [BKR95], pp. 467–468), a set $A \subseteq [0, 1]$ is an N-set if and only if there are non-negative reals $(\rho_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} \rho_n = \infty$ such that the series $\sum_{n=1}^{\infty} \rho_n \sin \pi nx$ absolutely converges for $x \in A$; if and only if there are reals $\rho_n \geq 0, k_n \geq 1, n = 1, 2, \ldots$ such that $\sum_{n=1}^{\infty} \rho_n = \infty$ and the series $\sum_{n=1}^{\infty} \rho_n \sin k_n x$ converges for $x \in E$. Every countable set is of type N. If E is of type N and D countable, then $E + D$ is of type N. Marcinkiewicz [M38d] proved that we can not replace D by an arbitrary set of type N, that is, the sum of two N-sets is not an N-set in general. The sum of two sets of type N may be the whole interval $(0, 2\pi)$, and so not of type N.

THEOREM 32 (Marcinkiewicz 1938). *There exist two sets* A *and* B *of type* N *such that* $A + B$ *is not of type* N*.*

A proof of this theorem we can find, e.g. in the Zygmund book ([Zy59], I, pp. 238–239) and Bary book ([Ba64], II, pp. 305–306). Arbault gave in [Ar52] a simple proof of this theorem.

[KS63] J.-P. Kahane and R. Salem, *Ensembles Parfaits et Séries Trigonométriques*, Hermann, Paris 1963; 2nd ed., Paris 1994.

[Ar52] J. Arbault, *Sur l'ensemble de convergence absolue d'une série trigonométrique*, Bull. Soc. Math. France 80(1952), 253–317.

[BKR95] L. Bukovský, N. N. Kholshchevnikova, and M. Repický, *Thin sets of harmonic analysis and infinite combinatorics*, Real Anal. Exchange 20(1994/95), no. 2, 454–509.

[Ka02] J.-P. Kahane, *Sets of uniqueness and sets of multiplicity*, Banach Center Publ. 56 (2002), 55–68 [4. Zygmund and Marcinkiewicz, p. 58].

[Sa41] R. Salem, *On some properties of symmetrical perfect sets*, Bull. Amer. Math. Soc. 47(1941), 820–828.

[Ul02] P. L. Ulyanov, *On interconnections between the research of Russian and Polish mathematicians in the theory of functions*, Banach Center Publ. 56(2002), 119–130.

4.6.4. *The Lévy–Raikov–Marcinkiewicz theorem on analytic properties of characteristic functions..* The following principle is well known in harmonic analysis: if μ is a positive finite measure and its Fourier transform $\hat{\mu}(t) = \int_{\mathbb{R}} e^{-itx}\, d\mu(x)$ is "smooth" at the origin, then it is "smooth" on the whole real line.

P. Lévy (1937) and D. Raikov (1938) (see, e.g., [LO77], Thm 2.2.1, p. 24) showed that if the Fourier transform $\hat{\mu}$ coincides in the real neighbourhood $(-a, a)$ of the origin with a function which is analytic in a rectangle $\{z : |\mathrm{Re} z| < a, -R < \mathrm{Im} z < R\}$, then $\hat{\mu}$ admits analytic continuation to the strip $\{z : |\mathrm{Im}| < R\}$. As a generalization of the real analyticity in $(-a, a) \subset \mathbb{R}$, one can consider weaker property of a function g to be the boundary value of a function which is analytic in a complex upper half-neighbourhood of $(-a, 0)$:

(A) g coincides in a real neighbourhood $(-a, a)$ of the origin with a function which is analytic in a rectangle $\{z : |\mathrm{Re} z| < a, 0 < \mathrm{Im} z < R\}$ and continuous in its closure.

Marcinkiewicz [M38e] (see also [LO77], Thm 2.2.3, p. 25) showed that the principle also works with this generalized real analyticity. The result can be stated in the following form:

The Lévy–Raikov–Marcinkiewicz theorem (1937–1938): *if μ is a finite nonnegative Borel measure whose Fourier transform $\hat{\mu}$ satisfies assumption (A), $\hat{\mu}$ admits analytic continuation into the strip $\{z : 0 < \mathrm{Im}\, z < R\}$ and is representable there by the absolutely convergent integral.*

The extension of this result to general classes of measures and distributions, assuming non-negativity only on some half-line $(b, +\infty)$ were given by Ostrovskii and Ulanovskii in [OU03] and [OU04].

[Ka72] T. Kawata, *Fourier Analysis in Probability Theory*, Academic Press, New York-London 1972 [theorem given by Marcinkiewicz, pp. 456–457].

[LO77] Ju. V. Linnik and I. V. Ostrovskii, *Decomposition of Random Variables and Vectors*, AMS, Providence 1977 [Theorem 2.2.3. Marcinkiewicz theorem, pp. 25–27 and 360].

[OU03] I. Ostrovskii and A. Ulanovskii, *On the Lévy–Raikov–Marcinkiewicz theorem*, C. R. Math. Acad. Sci. Paris 336(2003), no. 3, 237–240.

[OU04] I. Ostrovskii and A. Ulanovskii, *On the Lévy–Raikov–Marcinkiewicz theorem*, J. Math. Anal. Appl. 296(2004), no. 1, 314–325.

4.6.5. *The circular structure of the set of limit points of partial sums of Taylor series.* For a complex power series $\sum a_n z^n$, let $S_n(z) = \sum_{k=1}^{n} a_k z^k$, $s_n(x) := S_n(e^{ix})$ for $x \in \mathbb{R}$, and $\sigma_n(x) = \frac{s_0(x)+\cdots+s_n(x)}{n+1}$ (the $(C,1)$-mean). For each x, let $L(x)$ denote the set of limit points of $\{\sigma_n(x)\}$. Marcinkiewicz and Zygmund [MZ41] proved the following theorem (see also Zygmund [Zy59], II, pp. 178–179): *If $E := \{x : \lim_{n\to\infty} \sigma_n(x) = \sigma(x)$ exists$\}$ then, for almost all $x \in E$ and for every $\alpha \in L(x)$, the whole circumference $\{z\colon\ |z - \sigma(x)| = |\alpha - \sigma(x)|\}$ is included in $L(x)$.*

The question of the angular equidistribution of $\{s_n(x)\}$ almost everywhere in E was considered by Kahane [Ka83]. Examples suggest that not only the limit points of $\{\sigma_n(x)\}$ but the partial sums $s_n(x)$ themselves lie on $L(x)$, and Katsoprinakis [Ka89] proved a general result in this direction when E is uncountable. A related result by Katsoprinakis and Nestoridis [KN89] also answers a question posed by Kahane. A final result is due to Nestoridis [Ne92] and concerns power series $\sum c_k z^k$ with $\liminf |c_k| > 0$ and E an infinite subset of the unit circle (see also [NP90]). In the paper [KP99] there is an extension of Marcinkiewicz–Zygmund $(C,1)$ result into (C,k) summability with $k \geq 1$.

[Ka83] J.-P. Kahane, *Sur la structure circulaire des ensembles de points limites des sommes partielles d'une série de Taylor*, Acta Sci. Math. (Szeged) 45(1983), no. 1–4, 247–251. 30B30 (30B10)

[Ka89] E. S. Katsoprinakis, *On a theorem of Marcinkiewicz and Zygmund for Taylor series*, Ark. Mat. 27(1989), no. 1, 105–126.

[KP99] E. S. Katsoprinakis and M. Papadimitrakis, *Extensions of a theorem of Marcinkiewicz–Zygmund and of Rogosinski's formula and an application to universal Taylor series*, Proc. Amer. Math. Soc. 127(1999), no. 7, 2083–2090.

[Ne92] V. Nestoridis, *Limit points of partial sums of Taylor series*, Mathematika 38(1991), no. 2, 239–249 (1992).

[NP90] V. Nestoridis and S. K. Pichorides, *The circular structure of the set of limit points of partial sums of Taylor series*, Séminaire d'Analyse Harmonique, Année 1989/90, Univ. Paris XI, Orsay 1990, 71–77.

4.6.6. *Correction theorem.* Let $X_0 \subset X$ be two function spaces on $[0,1]$ with the Lebesgue measure m. Recall that X_0 is said to *correct* X if for every $f \in X$ and any $\varepsilon > 0$ there exists a function $g = g_\varepsilon \in X_0$ such that $m\{t \in [0,1] : f(t) \neq g(t)\} < \varepsilon$. The classical N. N. Luzin theorem (1912) on C-property of a measurable function $f\colon [a,b] \to \mathbb{R}$ states that for any $\varepsilon > 0$ there exists a closed set $F \subset [0,1]$ such that $m([0,1] \setminus F) < \varepsilon$ and $f_{|F}$ is continuous. This means that $C[0,1]$ correct $L^0[0,1]$–measurable functions on $[0,1]$. In other words, a measurable function we can correct on arbitrary small set in such a way that the function is continuous. Marcinkiewicz showed that ([M36a], Thm 3): $C^1[0,1]$ *correct* $Lip^1[0,1]$. Marcinkiewicz, in fact, assumed "pointwise" Lip^1, that is, $f(x+t) = f(x) + O(t)$ for any x and showed similar results for higher order smoothness. Federer [Fe44] obtained the analogous $Lip^1 - C^1$ result in higher dimension, and Whitney [Wh51] extended Federer's result to higher order of smoothness. He also gave an example of one variable function $\phi \in Lip^\alpha[0,1]$ for any $0 < \alpha < 1$ for which conclusion in Marcinkiewicz theorem is not true for it. Since we have $C^1 \subset Lip^1 \subset \cap_{0<\alpha<1} Lip^\alpha$, then Whitney example means that one could not weaken the requirement of f being in Lip^1 in the Marcinkiewicz theorem by $f \in \cap_{0<\alpha<1} Lip^\alpha$. In [BK03] the authors showed that the Takagi–van der Waerden function is also such an example.

[BK03] J. B. Brown and G. Kozlowski, *Smooth interpolation, Hölder continuity, and the Takagi–van der Waerden function*, Amer. Math. Monthly 110(2003), no. 2, 142–147.

[Fe44] H. Federer, *Surface area II*, Trans. Amer. Math. Soc. 55(1944) 438–456.

[Ki95] S. V. Kisliakov, *A new correction theorem*, Izv. Akad. Nauk SSSR Ser. Mat. 48(1984), no. 2, 305–330; English transl. in Math. USSR.-Izv. 24(1985), no. 2, 283–306.

[Ki95] S. V. Kisliakov, *A sharp correction theorem*, Studia Math. 113(1995), no. 2, 177–196.

[Wh51] H. Whitney, *On totally differentiable and smooth functions*, Pacific J. Math. 1(1951) 143–159.

Marcinkiewicz proved also many other theorems, but they did not have that much resonance as those previously presented.

Acknowledgments. I wish to thank Archive of the Institute of Mathematics Polish Academy of Sciences in Sopot, and especially Mrs. Janina Potocka–Schwartz, for xerox-copies, scans of documents and photos of Józef Marcinkiewicz. I thank Mrs. Emilia Jakimowicz from the University of Gdańsk for the exchange of information, photos made in the last few years and motivation to this work. I thank Mrs. Halina Kalinowska – daughter of Mieczysław Marcinkiewicz, Irena Makowska from Warsaw for sharing family photos and information about the family, and giving me the manuscript written by Mieczysław Marcinkiewicz [MM76], Mrs. Maria Lewicka from Białystok for the photo of Marcinkiewicz in military uniform, Prof. Romuald Brazis from the Polish University in Vilnius for the article about Marcinkiewicz and assistance during my stay in Vilnius in March 2010. In addition, words of thanks go to Dr. Andrzej Czesław Żak from the Central Military Archive in Rembertów for sending a xerox-copy of some documents on Marcinkiewicz and to Mrs. Janina Marciak-Kozłowska from the Institute of Electron Technology in Warsaw for exchange information and initiative to give the school in Janów the name of Józef Marcinkiewicz, as it was done on 14 October 2010. Thank also to colleagues Prof. Lars-Erik Persson (Luleå), Dr. Mark Rayson (Luleå), Prof. Sergei V. Astashkin (Samara), Karol Leśnik (Poznań) and Prof. Witold Wnuk (Poznań) for careful reading of the first version of this paper and for corrections.

The photos enclosed here are coming from the Marcinkiewicz's family collections (photos 2, 8, 10, 11), Archive of the Institute of Mathematics Polish Academy of Sciences in Sopot (photos 1, 6, 15), Central Lithuanian Archive in Vilnius (photos 3, 4, 5, 7), Library of the Nicolaus Copernicus University in Toruń (photos 9, 14). I want to thank all of them for permition to put these photos here. Photos 12 and 13 I did myself in 2010.

List of publications of Józef Marcinkiewicz[14]

[M33] *A new proof of a theorem on Fourier series*, J. London Math. Soc. **8**(1933), 179 [JMCP, 35].

[M34] *On a class of functions and their Fourier series*, C. R. Soc. Sci. Varsovie **26**(1934), 71–77 [JMCP, 36–41].

[14] In square brackets are given the pages on which the paper has been reprinted or translated in [JMCP] = *Józef Marcinkiewicz, Collected Papers*, Edited by Antoni Zygmund, PWN, Warsaw 1964, viii+673 pp.

[M35a] *Sur les nombres dérivés*, Fund. Math. **24**(1935), 305–308 [JMCP, 42–44].

[M35b] *Wielomiany interpolacyjne funkcji bezwzględnie ciągłych* [*Interpolating polynomials for absolutely continuous functions*], PhD Thesis, Universitas Vilnensis Batorena, Facultas Scientarum–Dissertationes Inaugurales, No. 10, Warszawa 1935, 1–35 (Polish) [copy of this dissertation is, for example, in Archive [MJ1]).

[M35c] *Wielomiany interpolacyjne funkcji bezwzględnie ciągłych* [*Interpolating polynomials for absolutely continuous functions*], Wiadom. Mat. **39**(1935), 85–125 (Polish); English transl. in [JMCP], 45–70.

[M35d] *On the convergence of Fourier series*, J. London Math. Soc. **10**(1935), 264–268 [JMCP, 71–75].

[M35e] *On Riemann's two methods of summation*, J. London Math. Soc. **10**(1935), 268–272 [JMCP, 76–80].

[MJZ35] (and B. Jessen, A. Zygmund) *Note on the differentiability of multiple integrals*, Fund. Math. **25**(1935), 217–234 [JMCP, 81–95].

[M36a] *Sur les séries de Fourier*, Fund. Math. **27**(1936), 38–69 [JMCP, 96–124].

[MZ36] (and A. Zygmund) *On the differentiability of functions and summability of trigonometrical series*, Fund. Math. **26**(1936), 1–43 [JMCP, 125–163].

[M36b] *Sur l'interpolation I*, Studia Math. **6**(1936), 1–17 [JMCP, 171–185].

[M36c] *Sur la convergence des séries orthogonales*, Studia Math. **6**(1936), 39–45 [JMCP, 164–170].

[M36d] *Sur l'interpolation II*, Studia Math. **6**(1936), 67–81 [JMCP, 186–199].

[M36e] *Quelques remarques sur l'interpolation*, Acta Litt. Sci. Szeged, Sect. Sci. Math. **8** (1936/37), 127–130 [JMCP, 200–203].

[M36f] *Sur la divergence des polynomes d'interpolation*, Acta Litt. Sci. Szeged, Sect. Sci. Math. **8**(1936/37), 131–135 [JMCP, 204–208].

[MZ37a] (and A. Zygmund) *Mean values of trigonometrical polynomials*, Fund. Math. **28** (1937), 131–166 [JMCP, 233–259].

[MZ37b] (and A. Zygmund) *Some theorems on orthogonal systems*, Fund. Math. **28**(1937), 309–335 [JMCP, 209–232].

[MZ37c] (and A. Zygmund) *Sur les fonctions indépendantes*, Fund. Math. **29**(1937), 60–90 [JMCP, 233–259].

[MZ37d] (and A. Zygmund) *Two theorems on trigonometric series*, Rec. Math. Moscou (Mat. Sbornik) **2**(44)(1937), 733–737 [JMCP, 293–298].

[MZ37e] (and A. Zygmund) *Remarque sur la loi logarithme itéré*, Fund. Math. **29**(1937), 215–222 [JMCP, 299–306].

[M37a] *Quelques théorèmes sur les séries orthogonales*, Ann. Soc. Polon. Math. **16**(1937), 84–96 [JMCP, 307–318].

[M37b][15] *Quelques théorèmes sur les séries et les fonctions*, Trav. Soc. Sci. Lettres Wilno, Cl. Sci. Math. Nat. **12**(1937), 1–6.

[15]R. Salem [Sa41] (see references in Part 4.6.3) refered to this Marcinkiewicz paper [M37b] which is not on the list of publications published in [JMCP]. Moreover, A. Zygmund in his review of Salem's paper in Math. Reviews MR0005132 (3,105f) added even pages on which Marcinkiewicz's article was published and this information was not in the Salem's article [Sa41]. Marcinkiewicz's paper [M37b] is also cited in the articles [Ar52] and [BKR95]. Paper has identical title as Marcinkiewicz's paper [M38f], but it is not clear if they are the same or different papers.

[M38a] *O sumowalności szeregów ortogonalnych* [*On the summability of orthogonal series*], Wiadom. Mat. **44**(1938), 5–16 (Polish); English transl. in [JMCP], 319–327.

[M38b] *Sur les fonctions indépendantes I*, Fund. Math. **30**(1938), 202–214 [JMCP, 328–340].

[M38c] *Sur les fonctions indépendantes II*, Fund. Math. **30**(1938), 349–364 [JMCP, 341–354].

[M38d] *Sur les suites d'opérations linéaires*, Studia Math. **7**(1938), 52–72 [JMCP, 355–373].

[MK38] (and S. Kaczmarz) *Sur les multiplicateurs des séries orthogonales*, Studia Math. **7**(1938), 73–81 [JMCP, 389–396].

[MZ38a] (and A. Zygmund) *Quelques théorèmes sur les fonctions indépendantes*, Studia Math. **7**(1938), 104–120 [JMCP, 374–388].

[M38e] *Sur les fonctions indépendantes III*, Fund. Math. **31**(1938), 86–102 [JMCP, 397–412].

[M38f] *Quelques théorèmes sur les séries et les fonctions*, Bull. Sémin. Math. Univ. Wilno **1**(1938), 19–24 [JMCP, 413–417].

[M38g] *Quelques théorèmes sur les séries orthogonales lacuinaires*, Ann. Soc. Polon. Math. **17**(1938), 51–58 [JMCP, 418–423].

[MZ38b] (and A. Zygmund) *Proof of a gap theorem*, Duke Math. J. **4**(1938), no. 3, 469–472 [JMCP, 424–427].

[MZ38c] (and A. Zygmund) *A theorem of Lusin*, Duke Math. J. **4**(1938), no. 3, 473–485 [JMCP, 428–443].

[M38h] *Sur quelques intégrales du type de Dini*, Ann. Soc. Polon. Math. **17**(1938), 42–50 [JMCP, 444–451].

[MZ38d] (and A. Zygmund) *Sur les séries de puissances*, Mathematica, Cluj **14**(1938), 21–30 [JMCP, 454–462].

[M38i] *Un théorème sur l'interpolation*, Mathematica, Cluj **14**(1938), 36–38 [JMCP, 452–453].

[M39a] *Sur une propriété de la loi de Gauss*, Math. Z. **44**(1939), no. 1, 612–618 [JMCP, 463–469].

[M39b] *Une remarque sur les espaces de M. Besicowitch*, C. R. Acad. Sci. Paris **208**(1939), 157–159 [JMCP, 470–472].

[M39c] *Sur le problème des moments*, C. R. Acad. Sci. Paris **208**(1939), 405–407 [JMCP, 473–474].

[M39d] *Sur la sommabilité H_k des séries de Fourier*, C. R. Acad. Sci. Paris **208**(1939), 782–784 [JMCP, 475–477].

[M39e] *Sur les séries orthogonales*, Studia Math. **8**(1939), 1–27 [JMCP, 478–500].

[M39f] *Sur les multiplicateurs des séries de Fourier*, Studia Math. **8**(1939), 78–91 [JMCP, 501–512].

[M38-40] *Sur une propriété du mouvement brownien*, Acta Litt. Sci. Szeged **9**(1938–40), 77–87 [JMCP, 513–523].

[MB39] (and S. Bergmann) *Sur les valeurs limites des fonctions de deux variables complexes*, C. R. Acad. Sci. Paris **208**(1939), 877–879 [JMCP, 524–526].

[M39g] *Sur une méthode remarquable de sommation des séries doubles de Fourier*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) **8**(1939), no. 2, 149–160 [JMCP, 527–538].

[M39h] *Sur l'interpolation d'opérations*, C. R. Acad. Sci. Paris **208**(1939), 1272–1273 [JMCP, 539–540].

[MZ39a] (and A. Zygmund) *Quelques inéqualitiés pour les opérations linéaires*, Fund. Math. **32**(1939), 115–121 [JMCP, 541–546].

[MZ39b] (and A. Zygmund) *On the summability of double Fourier series*, Fund. Math. **32** (1939), 122–132 [JMCP, 547–556].

[M39i] *Sur les variables aléatoires enroulées*, C. R. Soc. Math. France, année 1938(1939), 34–36 [JMCP, 557–558].

[M39j] *Sur la sommabilité forte de séries de Fourier*, J. London Math. Soc. **14**(1939), 162–168 [JMCP, 559–565].

[M39k][16] *Quelques théorèmes de la théorie des probabilités*, Bull. Sém. Math. Univ. Wilno **2**(1939), 22–34 [JMCP, 566–579].

[M39l] *Sur une nouvelle condition pour la convergence presque partout des séries de Fourier*, Ann. Scuola Norm. Super. Pisa (2) **8**(1939), 239–240 [JMCP, 580–581].

[MZ39c] (and A. Zygmund) *Sur la dérive seconde généralisée*, Bull. Sém. Math. Univ. Wilno **2**(1939), 35–40 [JMCP, 582–587].

[M40] *Sur la convergence absolue des séries de Fourier*, Mathematica, Cluj **16**(1940), 66–73 [JMCP, 588–594].

[MS40] (and R. Salem) *Sur une nouvelle condition pour la convergence presque partout des séries de Fourier*, Compositio Math. **7**(1940), 376–389 [JMCP, 595–608].

[MZ41] (and A. Zygmund) *On the behavior of trigonometric series and power series*, Trans. Amer. Math. Soc. **50**(1941), 407–453 [JMCP, 609–654].

[MB42] (and S. Bergmann) *Sur les fonctions analytiques de deux variables complexes*, J. Math. Phys. Mass. Inst. Tech. **21**(1942), 125–141 and Fund. Math. **33**(1945), 75–94 [JMCP, 655–673].

Information on Józef Marcinkiewicz

[Bi94] M. Bielski, *Józef Marcinkiewicz (1910–1940)*, in: "Polish Language Ancient and Modern", Materials from the National Conferences on Linguistics, ed. Cz. Łapicz, Nicholas Copernicus University Press, Toruń 1994, 155–172 (Polish).

[Br10] R. Brazis, *Mathematician Józef Marcinkiewicz – successor of the tradition of the Republic of Common Nations* (*for the 100th anniversary of birth*), Studium Vilnense A **7**(2010), 167–171 (Polish).

[DH95] K. Dąbrowski i E. Hensz, *Józef Marcinkiewicz (1910–1940) – on the 55th anniversary of death*, Polish Mathematical Society, Scientific Session, Toruń 13–15 Sept. 1995, 1–7 (Polish).

[DH02] K. Dąbrowski i E. H. Hensz-Chądzyńska, *Józef Marcinkiewicz (1910–1940) in commemoration of the 60th anniversary of his death*, in: "Fourier Analysis and Related Topics", Banach Center Publications 56, Warszawa 2002, 1–5.

[DP10] S. Domoradzki and Z. Pawlikowska-Brożek, *Józef Marcinkiewicz (1910–1940) in documents and memories: the centennial birthday*, Analecta **19**(2010), no. 1–2, 23–62 (Polish).

[Go92] Z. Godlewski, *Lived through Starobielsk*, Military Historical Overview **38**(1992), Z. 2, 306–331 (Polish).

[He92] E. Hensz, *Józef Marcinkiewicz (1910–1940) on the 80th anniversary of birth and 50th anniversary of death*, in: "Probability and Mechanics in the Historical Sketches", Proc. of the 5th All-Polish School on the History of Mathematics (Dziwnów, 9–13 May 1991), Edited by Stanisław Fudali, Part 1. Probability, Szczecin 1992, 253–258 (Polish).

[16] W. Feller in [Fe45] and [Fe46] (see references in Part 4.2.3) cited this Marcinkiewicz paper as J. Marcinkiewicz, *Quelques théorèmes de la théorie des probabilités*, Trav. Soc. Sci. Lettres Wilno, Cl. Sci. Math. Nat. **13**(1939), 1–13 (see also footnote 15).

[Ig05] S. Igari, *Józef Marcinkiewicz and real analysis in twentieth century*, Sugaku **57**(2005), no. 3, 302–309 (Japanese).

[Ig08] S. Igari, *Legacy of J. Marcinkiewicz to real analysis in the 20th century*, in: "Selected Papers on Analysis and Related Topics", Amer. Math. Soc. Transl. Ser. 2, 223, AMS, Providence 2008, 53–61.

[Je71] L. Jeśmanowicz, *Recollections of Vilnius mathematicians*, Wiadom. Mat. **12**(1971), no. 2, 309–319 (Polish).

[Je74] L. Jeśmanowicz, *Józef Marcinkiewicz*, Polish Biographical Dictionary **19**(1974), 587–588 (Polish).

[Ka01] S. Kalbarczyk, *Józef Marcinkiewicz*, in: "Polish research workers, victims of Soviet crime during the Second World War", NERITON Publishing House, Warszawa 2001, 132–133 (Polish).

[Ko73] S. Kolankowski, *Reminiscences on Józef Marcinkiewicz*, Wiadom. Mat. **16**(1973), 75–77 (Polish).

[Ko03] S. Kolankowski, *Józef Marcinkiewicz (1910–1940)*, in: "Biographical Dictionary of Polish Mathematicians", Ed. S. Domaradzki, Z. Pawlikowska-Brożek, D. Węglowska, Tarnobrzeg 2003, p. 149 (Polish).

[Ko92] B. Koszela, *The contribution of Józef Marcinkiewicz, Stefan Mazurkiewicz and Hugo Steinhaus in developing Polish mathematics. A biographical sketch*, in: "Mathematics at the Turn of the Twentieth Century" (Jawor, 1988), Prace Nauk. Uniw. Śląsk. Katowic. 1253, Uniw. Śląski, Katowice 1992, 104–110 (Polish).

[Le59] S. Lewicka, *Letter of Stanisława Lewicka – sister of Józef Marcinkiewicz to "Mathematical News" dated 12 January 1959 from Sokółka*, Archive of the Institute of Mathematics Polish Academy of Sciences in Sopot, 4 pages (Polish).

[Ma10a] L. Maligranda, *Józef Marcinkiewicz and his mathematical achievements*, talk at the 24th All-Polish School on the History of Mathematics, Iwonicz Zdrój (24–28 May 2010) given on 25 May 2010, 68 pages (Polish).

[Ma10b] L. Maligranda, *Józef Marcinkiewicz (1910–1940) – on the centenary of his birth*, plenary lecture at the international conference *The Józef Marcinkiewicz Centenary Conference (JM100)*, Poznań, 28 June–2 July 2010, 62 pp. + 44 pp. on his mathematics.

[Ma10c] L. Maligranda, *Józef Marcinkiewicz – extraordinary mathematical talent*, Luleå, 5 Dec. 2010, 1–72 (Polish).

[MJ1] Józef Marcinkiewicz, archive material in: Lietuvos Centrinis Valstybės Archyvas, Fond 175, Ap. VII B, b. 165, 216, 267, 281; Ap. VIIDb, b. 216; Ap. I Bb, p. 875; Ap. XIV, b. 260.

[MJ2] Józef Marcinkiewicz, materials in the Central Military Archive in Rembertów (CAW, ap. Józef Marcinkiewicz nr 10342).

[JMCP] *Józef Marcinkiewicz, Collected Papers*, Edited by A. Zygmund. With the collaboration of S. Łojasiewicz, J. Musielak, K. Urbanik and A. Wiweger, IMPAN, PWN, Warsaw 1964, viii+673 pp.

[MM76] M. Marcinkiewicz, *Description of the Marcinkiewicz family*, 1976, manuscript, 60 pages (Polish).

[OR] J. O'Connor and E. F. Robertson, *Józef Marcinkiewicz*, at: http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Marcinkiewicz.html

[Zy60] A. Zygmund, *Józef Marcinkiewicz*, Wiadom. Mat. **4**(1960), 11–41 (Polish).

[Zy64] A. Zygmund, *Józef Marcinkiewicz*, in: *Józef Marcinkiewicz, Collected Papers*, PWN, Warsaw 1964, 1–33.

References

[Ba32] S. Banach, *Théorie des Opérations Linéaires*, Monografie Matematyczne 1, Warszawa 1932.

[Bi00] N. H. Bingham, *Studies in the history of probability and statistics XLVI. Measure into probability: From Lebesgue to Kolmogorov*, Biometrika **87**(2000), no. 1, 145–156.

[Ca83] A. P. Calderón, *Antoni Zygmund*, in: "Conference on Harmonic Analysis in Honor of Antoni Zygmund" held at the University of Chicago, Chicago, Ill., March 23–28, 1981, Edited by W. Beckner, A. P. Calderón, R. Fefferman and P. W. Jones. Wadsworth, Belmont 1983, Vol. I, xiii–xv.

[DP99] S. Domoradzki and Z. Pawlikowska-Brożek, *Vilnius University*, Wiadom. Mat. **35**(1999), 133–139 (Polish).

[DP00] S. Domoradzki and Z. Pawlikowska-Brożek, *Vilnius between the wars*, Math. Intelligencer **22**(2000), no. 4, 47–50.

[Du07] R. Duda, *The Lvov School of Mathematics*, University of Wrocław Publisher, Wrocław 2007 (Polish).

[DEMP] P. Dugac, B. Eckman, J. Mawhin and J.-P. Pier, *Guidelines 1900–1950*, in: "Development of Mathematics 1900–1950", Lectures from the symposium held in Luxembourg, June 1992, Edited by Jean-Paul Pier, Birkhäuser, Basel 1994, 1–34.

[FKS76] Ch. Fefferman, J.-P. Kahane i E. M. Stein, *The scientific output of Antoni Zygmund*, Wiadom. Mat. **19**(1976), no. 2, 91–126 (Polish); English version *A review of A. Zygmund's scientific work*, in: "Selected Papers of Antoni Zygmund", Edited by A. Hulanicki, P. Wojtaszczyk and W. Żelazko, Kluwer, Dordrecht 1989, Vol. I, xix–xlviii.

[Ho97] P. Holgate, *Independent functions: Probability and analysis in Poland between the Wars*, Biometrika **84**(1997), no. 1, 161–173.

[Ho09] J. Horváth, *Encounters with Mischa Cotlar*, Notices Amer. Math. Soc. **56**(2009), no. 5, 616–620.

[Ku73] K. Kuratowski, *A half century of Polish mathematics. Remembrances and reflections*, Wiedza Powszechna, Warszawa 1973 (Polish); English transl. PWN and Pergamon Press, Warszawa 1980.

[Ku81] K. Kuratowski, *Notes for an autobiography*, Czytelnik, Warszawa 1981 (Polish).

[Le39] P. Lévy, *Sur un problème de M. Marcinkiewicz*, C. R. Acad. Sci. Paris **208**(1939), 318–321.

[Mau81] R. D. Mauldin, *The Scottish Book. Mathematics from the Scottish Café*, Birkhäuser, Boston 1981.

[Or81] W. Orlicz, *The Lvov School of Mathematics in the interwar period*, Wiadom. Mat. **23**(1981), no. 2, 222–231 (Polish).

[Or88] W. Orlicz, *Achievements of Polish mathematicians in the domain of functional analysis in the years 1919–1951*, in: "Władysław Orlicz, Collected Papers", PWN Polish Scientific Publishers, Warsaw 1988, 1616–1641.

[Pe02] J. Peetre, *On the development of interpolation – instead of a history three letters*, in: "Function Spaces, Interpolation Theory and Related Topics", Proc. of the International Conference in honour of Jaak Peetre on his 65th birthday (held at Lund University, Lund, Aug. 17–22, 2000), Edited by M. Cwikel, M. Englis, A. Kufner, L.-E. Persson and G. Sparr, Walter de Gruyter, Berlin 2002, 39–48.

[Re04] C. B. Read, *A conversation with Norman L. Johnson*, Statist. Sci. **19**(2004), no. 3, 544–560.

[Sa01] C. Sadosky, *Look again at Vilnius: "Vilnius between the wars" [Math. Intelligencer 22 (2000), no. 4, 47–50] by S. Domoradzki and Z. Pawlikowska-Brożek*, Math. Intelligencer **23**(2001), no. 1, 5–6.

[St83] E. M. Stein, *The development of square functions in the work of A. Zygmund*, in: "Conference on Harmonic Analysis in Honor of Antoni Zygmund" held at the University of Chicago, Chicago, Ill., March 23–28, 1981, Edited by W. Beckner, A. P. Calderón, R. Fefferman and P. W. Jones. Wadsworth, Belmont 1983, Vol. I, 2–30.

[Ta01] M. S. Taqqu, *Bachelier and his times: a conversation with Bernard Bru*, Finance Stoch. **5**(2001), no. 1, 3–32 [Marcinkiewicz, p. 24]; Reprinted in *Mathematical finance – Bachelier Congress 2000*, Selected papers from the 1st World Congress of the Bachelier Finance Society held in Paris, June 29–July 1, 2000, Edited by H. German, D. Madan, S. R. Pliska and T. Vorst, Springer, Berlin 2002, 1–39 [Marcinkiewicz, p. 23].

[Tu55] P. Turán, *The life and mathematics of Géza Grünwald*, 1 April 1955, 4 pages at: `http://www.focm.net/at/HAT/grunwald.html`

[Zy51] A. Zygmund, *Polish mathematics between the two wars (1919–1939)*, in: "Proc. Second Canadian Math. Congress" (Vancouver 1949), University of Toronto Press, Toronto 1951, 3–9.