

# ON CONTRACTION PRINCIPLE APPLIED TO NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH DERIVATIVES OF ORDER $\alpha \in (0, 1)$

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**Abstract.** One-term and multi-term fractional differential equations with a basic derivative of order  $\alpha \in (0, 1)$  are solved. The existence and uniqueness of the solution is proved using the fixed point theorem and the equivalent norms designed for a given value of parameters and function space. The explicit form of the solution obeying the set of initial conditions is given.

**1. Introduction.** Fractional differential equations are an important tool in the mathematical modelling of many systems and processes in mechanics, physics, chemistry, biochemistry, control theory, economics, engineering and bioengineering. Investigations on fractional differential equations include solving methods, the existence and uniqueness of solutions and studies of the properties of solutions as well as their applications. During last decades they yielded many essential results and theory of fractional differential equations became an important part of pure and applied mathematics (compare monographs and review papers [5, 6, 7, 9, 10, 13, 15, 16, 17, 18, 24] and the references therein). As mentioned in monographs [7] and [17] fractional differential equations of higher order include a class of sequential fractional differential equations. Such equations are applied for instance in hydrodynamics [4, 21, 22] and in theory of viscoelasticity [23]. Our aim is to study a variation of sequential fractional differential equations with composed differential operator including derivatives of a given order and a variable coefficient.

In the paper we consider nonlinear multi-term fractional differential equations dependent on the basic fractional derivative of arbitrary real noninteger order  $\alpha \in (0, 1)$ . The other types of sequential fractional differential equations were also studied in [7, 8, 19, 25].

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To prove the existence and uniqueness of the solutions in an arbitrary finite interval we follow the fixed point method and apply the Banach theorem. Our aim is to propose an efficient method of proof. A crucial point in the proof is the application of a newly-introduced one-parameter equivalent norms (and respective metrics) in the space of continuous or continuous weighted functions.

The paper is organized as follows. In the next section we recall all the necessary definitions and properties of fractional operators. We also construct one-parameter families of equivalent norms and respective metrics in the space of continuous and weighted continuous functions in a finite interval. Then we prove that certain fractional integral operators are bounded in these spaces endowed with a corresponding norm from the proposed class. The basic integral operator is generalized to a mapping, which appears to be a contraction under the respective assumptions on a parameter defining the norm and metric on the functions space. Section 3 contains the main results - theorems on the existence and uniqueness of the solution to a certain nonlinear one-term and multi-term fractional differential equations. The paper is closed with a short discussion of the results and their prospective extension to further types of fractional differential equations.

**2. Preliminaries.** In the paper we shall consider solutions of certain class of fractional differential equations in the space of functions continuous in finite interval  $[0, b]$ . The supremum norm on the  $C[0, b]$  space and the respective induced metric are given below:

$$\|f\| := \sup_{t \in [0, b]} |f(t)| \quad d(f, g) := \|f - g\|. \quad (1)$$

The norm  $\|\cdot\|_\gamma$  and the generated metric are active in the space of weighted continuous functions, when  $Re(\gamma) \in (0, 1)$ :

$$\|f\|_\gamma := \sup_{t \in [0, b]} |t^\gamma f(t)| \quad d'(f, g) := \|f - g\|_\gamma. \quad (2)$$

The  $C_\gamma[0, b]$  space is then given as

$$C_\gamma[0, b] := \{f \in C(0, b]; \|f\|_\gamma \leq \infty\}. \quad (3)$$

REMARK. Both metric functions spaces  $\langle C[0, b], d \rangle$  and  $\langle C_\gamma[0, b], d' \rangle$  are complete.

Now, we recall definitions of left-sided fractional operators. In our paper we shall study fractional differential equations containing Riemann-Liouville or Caputo derivatives. Both, the integral and derivatives, are defined as follows [7, 20].

DEFINITION 2.1. Let  $Re(\alpha) > 0$ . Then the left-sided Riemann-Liouville integral of order  $\alpha$  is given by the formula

$$(I_{0+}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0, \quad (4)$$

where  $\Gamma$  denotes the Euler gamma function.

Let  $Re(\alpha) \in (n-1, n)$ . Then the left-sided Riemann-Liouville derivative is defined as

$$(D_{0+}^\alpha f)(t) = \left(\frac{d}{dt}\right)^n (I_{0+}^{n-\alpha} f)(t) \quad t > 0. \quad (5)$$

Let  $Re(\alpha) \in (n - 1, n)$ . Then the left-sided Caputo derivative is defined as

$$({}^c D_{0+}^\alpha f)(t) = D_{0+}^\alpha \left[ f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)t^k}{k!} \right] \quad t > 0. \quad (6)$$

REMARK. In the paper we shall solve equations dependent on fractional derivatives of real order  $\alpha \in (0, 1)$ . In this case fractional derivatives are defined by the formulas below:

$$(D_{0+}^\alpha f)(t) = \frac{d}{dt}(I_{0+}^{1-\alpha} f)(t) \quad (7)$$

$$({}^c D_{0+}^\alpha f)(t) = D_{0+}^\alpha [f(t) - f(0)]. \quad (8)$$

An important and characteristic feature of the above fractional operators is their composition rule [7, 20]. It will be applied in the transformation of the investigated equations into their equivalent integral form as well as in the derivation of the corresponding initial conditions.

PROPERTY 2.2. *Let  $Re(\delta) > Re(\alpha) > 0$ . Then the following formulas*

$$D_{0+}^\alpha I_{0+}^\delta f(t) = I_{0+}^{\delta-\alpha} f(t) \quad (9)$$

$${}^c D_{0+}^\alpha I_{0+}^\delta f(t) = I_{0+}^{\delta-\alpha} f(t) \quad (10)$$

*hold at any point  $t \in [0, b]$  when function  $f \in C[0, b]$ . If  $f \in C_\gamma[0, b]$ , then the above composition rules hold at any point  $t \in (0, b]$ .*

When orders of the fractional derivative and integral coincide, we obtain the composition rules analogous to the second main theorem of integral calculus.

PROPERTY 2.3. *Let  $Re(\alpha) > 0$ . Then the following formulas*

$$D_{0+}^\alpha I_{0+}^\alpha f(t) = f(t) \quad (11)$$

$${}^c D_{0+}^\alpha I_{0+}^\alpha f(t) = f(t) \quad (12)$$

*hold at any point  $t \in [0, b]$  when function  $f \in C[0, b]$ . If  $f \in C_\gamma[0, b]$ , then the above composition rules hold at any point  $t \in (0, b]$ .*

In the procedure of transforming the considered fractional differential equation into an equivalent integral equation we shall apply the stationary functions of Riemann-Liouville or Caputo derivative. They are analogues of polynomial functions from classical calculus and differential equations theory. For order  $\alpha \in (0, 1)$  the continuous stationary function of Caputo derivative is an arbitrary constant:

$${}^c D_{0+}^\alpha \phi_0(t) = 0 \quad \forall t \in [0, b] \quad \iff \quad \phi_0(t) = c.$$

Respectively, the only continuous weighted stationary functions of Riemann-Liouville derivative are proportional to power function:

$$D_{0+}^\alpha \phi_0(t) = 0 \quad \forall t \in (0, b] \quad \iff \quad \phi_0(t) = ct^{\alpha-1}$$

and belong to the  $C_{1-\alpha}[0, b]$  space.

In what follows we shall modify norms (1,2) to the equivalent ones on the  $C[0, b]$  and  $C_{1-\alpha}[0, b]$  spaces respectively. To this aim we apply functions constructed using the Mittag-Leffler function [7]. We define auxiliary functions dependent on parameters  $\alpha, \beta$

and on free parameter  $\kappa$ , assuming  $\alpha \in (0, 1)$ ,  $\alpha - \beta > 0$ ,  $\kappa \in R_+$  and  $t \in (0, b]$ :

$$e^{\alpha, \beta, \kappa}(t) := \Gamma(\alpha - \beta) E_{\alpha, \alpha - \beta}(\kappa t^\alpha) \quad (13)$$

$$\mathcal{E}^{\alpha, \beta, \kappa}(t) := E_{\alpha - \beta, \alpha - \beta}(\kappa t^{\alpha - \beta}) \quad (14)$$

$$E^{\alpha, \beta, \kappa}(t) := \Gamma(1 - \beta) E_{\alpha, 1 - \beta}(\kappa t^\alpha) \quad (15)$$

$$\mathbf{E}^{\alpha, \beta, \kappa}(t) := E_{\alpha - \beta, 1 - \beta}(\kappa t^{\alpha - \beta}). \quad (16)$$

In the above formulas  $E_{A,B}$  is the two-parameter Mittag-Leffler function ( $Re(A) > 0$ ):

$$E_{A,B}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(Ak + B)}$$

determined (in general) on complex plane  $C$  for  $Re(A) > 0$ .

The following proposition describes integration properties of the introduced functions.

**PROPOSITION 2.4.** *If  $\alpha \in (0, 1)$ ,  $\alpha - \beta > 0$  and  $\kappa \in R_+$ , then the following formulas hold in any interval  $(0, b]$*

$$I_{0+}^\alpha t^{\alpha - \beta - 1} e^{\alpha, \beta, \kappa}(t) = \frac{t^{\alpha - \beta - 1}}{\kappa} [e^{\alpha, \beta, \kappa}(t) - 1] \quad (17)$$

$$I_{0+}^\alpha t^{\alpha - \beta - 1} \mathcal{E}^{\alpha, \beta, \kappa}(t) = \frac{t^{\alpha - 1}}{\kappa} \left[ E_{\alpha - \beta, \alpha}(\kappa t^{\alpha - \beta}) - \frac{1}{\Gamma(\alpha)} \right] \quad (18)$$

$$I_{0+}^\alpha t^{-\beta} E^{\alpha, \beta, \kappa}(t) = \frac{t^{-\beta}}{\kappa} [E^{\alpha, \beta, \kappa}(t) - 1]. \quad (19)$$

$$I_{0+}^\alpha t^{-\beta} \mathbf{E}^{\alpha, \beta, \kappa}(t) = \frac{1}{\kappa} [E_{\alpha - \beta, 1}(\kappa t^{\alpha - \beta}) - 1]. \quad (20)$$

Solving the considered fractional differential equations we shall follow the methods from differential equations theory. We start by transforming the fractional differential equation into its equivalent integral form and then into a fixed point condition for a mapping determined on the respective function space. To prove that this mapping is contractive we extend Bielecki method of equivalent norms [2]. Similar modification of norms was also developed in theory of fractional differential equations [1, 3]. Lakshmikantham et al [13, 14] used in the scaling procedure a one-parameter Mittag-Leffler function and proved the existence-uniqueness result for one-term fractional differential equation with Caputo derivative. We propose to apply in modification of norms the introduced in (13-16) auxiliary functions. Mappings, which are not contractive with respect to standard norms (1,2), become contractions after change of the norm and metric to the equivalent ones, dependent on free parameter  $\kappa \in R_+$ , provided  $\kappa$  is large enough.

**DEFINITION 2.5.** In the  $C_{1-\alpha}[0, b]$  space we define a new norm and metric, provided  $\kappa \in R_+$  and  $\alpha - \beta > 0$ :

$$\|f\|_{1-\alpha, \kappa} := \sup_{t \in [0, b]} \frac{|t^{1-\alpha} f(t)|}{e^{\alpha, \beta, \kappa}(t)} \quad d'_\kappa(f, g) := \|f - g\|_{1-\alpha, \kappa} \quad (21)$$

$$\|f\|'_{1-\alpha, \kappa} := \sup_{t \in [0, b]} \frac{|t^{1-\alpha} f(t)|}{\mathcal{E}^{\alpha, \beta, \kappa}(t)} \quad d''_\kappa(f, g) := \|f - g\|'_{1-\alpha, \kappa}. \quad (22)$$

In the  $C[0, b]$  space, we define a new norm and metric assuming  $\kappa \in R_+$  and  $\alpha - \beta > 0$  (see also [13, 14] for  $\beta = 0$ )

$$\|f\|_{\kappa} := \sup_{t \in [0, b]} \frac{|f(t)|}{E^{\alpha, \beta, \kappa}(t)} \quad d_{\kappa}(f, g) := \|f - g\|_{\kappa} \quad (23)$$

$$\|f\|'_{\kappa} := \sup_{t \in [0, b]} \frac{|f(t)|}{\mathbf{E}^{\alpha, \beta, \kappa}(t)} \quad d_{\kappa}^1(f, g) := \|f - g\|'_{\kappa} . \quad (24)$$

PROPOSITION 2.6. *Metrics  $d'_{\kappa}$  and  $d''_{\kappa}$  are equivalent to the  $d'$  metric in the  $C_{1-\alpha}[0, b]$  space. Metrics  $d_{\kappa}$  and  $d^1_{\kappa}$  are equivalent to the  $d$  metric in the  $C[0, b]$  space.*

*Proof.* The equivalence is implied by the set of inequalities fulfilled by the respective norms for arbitrary function  $f \in C_{1-\alpha}[0, b]$

$$\begin{aligned} \frac{\|f\|_{1-\alpha}}{e^{\alpha, \beta, \kappa}(b)} &\leq \|f\|_{1-\alpha, \kappa} \leq \|f\|_{1-\alpha} \\ \frac{\|f\|_{1-\alpha}}{\mathcal{E}^{\alpha, \beta, \kappa}(b)} &\leq \|f\|'_{1-\alpha, \kappa} \leq \Gamma(\alpha - \beta) \|f\|_{1-\alpha} \end{aligned}$$

and for arbitrary  $f \in C[0, b]$

$$\begin{aligned} \frac{\|f\|}{E^{\alpha, \beta, \kappa}(b)} &\leq \|f\|_{\kappa} \leq \|f\| \\ \frac{\|f\|}{\mathbf{E}^{\alpha, \beta, \kappa}(b)} &\leq \|f\|'_{\kappa} \leq \Gamma(1 - \beta) \|f\| . \end{aligned}$$

■

Having defined the new norms, we shall study the properties of integral operator  $I_{0+}^{\alpha} t^{-\beta}$  on the spaces of continuous and weighted continuous functions with norms (23,24) or (21,22) respectively. Analyzing the formulas enclosed in the proposition below we note that this operator is bounded in all the considered cases and that the constant on the right-hand side is inversely proportional to parameter  $\kappa$ .

PROPOSITION 2.7. *Let  $\alpha \in (0, 1)$ .*

(1) *If  $\beta \leq 0$ , then for any function  $f \in C_{1-\alpha}[0, b]$  the following inequalities hold ( $j \in N$ )*

$$\|(I_{0+}^{\alpha} t^{-\beta})^j f\|_{1-\alpha, \kappa} \leq \left(\frac{b^{-\beta}}{\kappa}\right)^j \|f\|_{1-\alpha, \kappa} . \quad (25)$$

(2) *If  $\beta > 0$  and  $\alpha - \beta > 0$ , then sequence  $(\kappa_m)$  exists such that  $\lim_{m \rightarrow \infty} \kappa_m = \infty$  and for any function  $f \in C_{1-\alpha}[0, b]$  the following inequalities hold ( $j \in N$ )*

$$\|(I_{0+}^{\alpha} t^{-\beta})^j f\|'_{1-\alpha, \kappa_m} \leq \left(\frac{A_L}{\kappa_m}\right)^j \|f\|'_{1-\alpha, \kappa_m} , \quad (26)$$

where constant  $L$  is determined by condition

$$(\alpha - \beta)(L + 1) \geq \gamma_{\min} \quad (\alpha - \beta)L < \gamma_{\min}$$

and constant  $A_L$  looks as follows

$$A_L = \sum_{k=1}^{L-1} \frac{\Gamma(\alpha - \beta)}{\Gamma(k(\alpha - \beta) + \alpha)} + 1.$$

(3) If  $\beta \leq 0$ , then for any function  $f \in C[0, b]$  the following inequalities hold ( $j \in N$ )

$$\| (I_{0+}^{\alpha} t^{-\beta})^j f \|_{\kappa} \leq \left( \frac{b^{-\beta}}{\kappa} \right)^j \| f \|_{\kappa}. \quad (27)$$

(4) If  $\beta > 0$  and  $\alpha - \beta > 0$ , then then sequence  $(\kappa_m)$  exists such that  $\lim_{m \rightarrow \infty} \kappa_m = \infty$  and for any function  $f \in C[0, b]$  the following inequalities hold ( $j \in N$ )

$$\| (I_{0+}^{\alpha} t^{-\beta})^j f \|'_{\kappa_m} \leq \left( \frac{B_L}{\kappa_m} \right)^j \| f \|'_{\kappa_m}, \quad (28)$$

where constant  $L$  is determined by condition:

$$(\alpha - \beta)L \geq \gamma_{\min} + \beta - 1 \quad (\alpha - \beta)(L - 1) < \gamma_{\min} + \beta - 1$$

and constant  $B_L$  looks as follows

$$B_L = \sum_{k=1}^{L-1} \frac{\Gamma(1 - \beta)}{\Gamma(k(\alpha - \beta) + 1)} + 1.$$

*Proof.* We begin with part (1) and observe that fractional integral  $I_{0+}^{\alpha}$  is bounded in the  $C_{1-\alpha}[0, b]$  space. Thus, it is enough to prove the case  $j = 1$  as  $j > 1$  follows from the mathematical induction principle. Applying integration property (17) from Proposition 2.4 we obtain for arbitrary function  $f \in C_{1-\alpha}[0, b]$ :

$$\begin{aligned} \| (I_{0+}^{\alpha} t^{-\beta}) f \|_{1-\alpha, \kappa} &= \sup_{t \in [0, b]} \frac{|t^{1-\alpha} I_{0+}^{\alpha} t^{-\beta} f(t)|}{e^{\alpha, \beta, \kappa}(t)} = \\ &= \sup_{t \in [0, b]} \frac{|t^{1-\alpha} I_{0+}^{\alpha} t^{-\beta} t^{\alpha-1} e^{\alpha, \beta, \kappa}(t) \frac{t^{1-\alpha} f(t)}{e^{\alpha, \beta, \kappa}(t)}|}{e^{\alpha, \beta, \kappa}(t)} \leq \\ &\leq \| f \|_{1-\alpha, \kappa} \sup_{t \in [0, b]} \frac{|t^{1-\alpha} I_{0+}^{\alpha} t^{-\beta} t^{\alpha-1} e^{\alpha, \beta, \kappa}(t)|}{e^{\alpha, \beta, \kappa}(t)} = \\ &= \| f \|_{1-\alpha, \kappa} \sup_{t \in [0, b]} \frac{t^{-\beta}}{\kappa} \left[ 1 - \frac{1}{e^{\alpha, \beta, \kappa}(t)} \right] \leq \\ &\leq \frac{b^{-\beta}}{\kappa} \cdot \| f \|_{1-\alpha, \kappa}. \end{aligned}$$

In the proof of part (2) we use formula (18) from Proposition 2.4 and monotonicity property of the Euler gamma function, namely for real argument greater than  $\gamma_{\min} \cong 1.46163$  this function is strictly increasing. We again prove the case  $j = 1$  as formulas for  $j > 1$  are a straightforward corollary. Let us observe that for any given  $\alpha$  and  $\beta$  fulfilling condition  $\alpha - \beta > 0$ , integer number  $L \in N$  exists such that

$$(\alpha - \beta)(L + 1) \geq \gamma_{\min} \quad (\alpha - \beta)L < \gamma_{\min}. \quad (29)$$

Let us denote

$$P_L(t) = \sum_{k=1}^{L-1} \frac{(\kappa t^{\alpha-\beta})^k}{\Gamma((\alpha - \beta)k + \alpha)} \quad R_L(t) = \sum_{k=L}^{\infty} \frac{(\kappa t^{\alpha-\beta})^k}{\Gamma((\alpha - \beta)k + \alpha)}$$

and

$$\mathbf{P}_L(t) = \sum_{k=1}^{L-1} \frac{(\kappa t^{\alpha-\beta})^k}{\Gamma((\alpha-\beta)k + \alpha - \beta)} \quad \mathbf{R}_L(t) = \sum_{k=L}^{\infty} \frac{(\kappa t^{\alpha-\beta})^k}{\Gamma((\alpha-\beta)k + \alpha - \beta)}.$$

It is easy to check that

$$\mathcal{E}^{\alpha,\beta,\kappa}(t) = \frac{1}{\Gamma(\alpha-\beta)} + \mathbf{P}_L(t) + \mathbf{R}_L(t).$$

We obtain the following inequalities for the  $\|\cdot\|'_{1-\alpha,\kappa}$  norm of integral  $I_{0+}^{\alpha} t^{-\beta} f$ :

$$\begin{aligned} \|(I_{0+}^{\alpha} t^{-\beta})f\|'_{1-\alpha,\kappa} &= \sup_{t \in [0,b]} \frac{|t^{1-\alpha} I_{0+}^{\alpha} t^{-\beta} f(t)|}{\mathcal{E}^{\alpha,\beta,\kappa}(t)} = \\ &= \sup_{t \in [0,b]} \frac{|t^{1-\alpha} I_{0+}^{\alpha} t^{-\beta} t^{\alpha-1} \mathcal{E}^{\alpha,\beta,\kappa}(t) \frac{t^{1-\alpha} f(t)}{\mathcal{E}^{\alpha,\beta,\kappa}(t)}|}{\mathcal{E}^{\alpha,\beta,\kappa}(t)} \leq \\ &\leq \|f\|'_{1-\alpha,\kappa} \sup_{t \in [0,b]} \frac{|t^{1-\alpha} I_{0+}^{\alpha} t^{-\beta} t^{\alpha-1} \mathcal{E}^{\alpha,\beta,\kappa}(t)|}{\mathcal{E}^{\alpha,\beta,\kappa}(t)} = \\ &= \|f\|'_{1-\alpha,\kappa} \sup_{t \in [0,b]} \frac{t^{1-\alpha} t^{\alpha-1} \left[ E_{\alpha-\beta,\alpha}(\kappa t^{\alpha-\beta}) - \frac{1}{\Gamma(\alpha)} \right]}{\kappa \cdot \mathcal{E}^{\alpha,\beta,\kappa}(t)} = \\ &= \frac{1}{\kappa} \cdot \|f\|'_{1-\alpha,\kappa} \sup_{t \in [0,b]} \frac{P_L(t) + R_L(t)}{\frac{1}{\Gamma(\alpha-\beta)} + \mathbf{P}_L(t) + \mathbf{R}_L(t)}. \end{aligned}$$

Thanks to assumption  $(\alpha-\beta)(L+1) \geq \gamma_{\min}$  we can rewrite the above inequality as follows:

$$\|(I_{0+}^{\alpha} t^{-\beta})f\|'_{1-\alpha,\kappa} \leq \frac{1}{\kappa} \cdot \|f\|'_{1-\alpha,\kappa} \sup_{t \in [0,b]} \left( \frac{P_L(t)}{\frac{1}{\Gamma(\alpha-\beta)} + \mathbf{P}_L(t)} + 1 \right).$$

In the above formula norm  $\|(I_{0+}^{\alpha} t^{-\beta})f\|'_{1-\alpha,\kappa}$  is estimated with a coefficient given as a supremum over interval  $[0, b]$  of certain rational function with positive denominator. As all these functions are continuous in  $[0, b]$ , the supremum is in fact a maximum - value of function at certain  $t_{\kappa} \in [0, b]$ :

$$\|(I_{0+}^{\alpha} t^{-\beta})f\|'_{1-\alpha,\kappa} \leq \frac{1}{\kappa} \cdot \|f\|'_{1-\alpha,\kappa} \left( \frac{P_L(t_{\kappa})}{\frac{1}{\Gamma(\alpha-\beta)} + \mathbf{P}_L(t_{\kappa})} + 1 \right).$$

Let us choose sequence  $(\kappa_m)$  such that

$$\lim_{m \rightarrow \infty} \kappa_m = \infty$$

and limit  $\lim_{m \rightarrow \infty} \kappa_m t_{\kappa_m}^{\alpha-\beta}$  (finite or infinite) exists. For this sequence of parameters we obtain the following sequence of inequalities:

$$\begin{aligned} \|(I_{0+}^{\alpha} t^{-\beta})f\|'_{1-\alpha,\kappa_m} &\leq \frac{1}{\kappa_m} \cdot \|f\|'_{1-\alpha,\kappa_m} \left( \frac{P_L(t_{\kappa_m})}{\frac{1}{\Gamma(\alpha-\beta)} + \mathbf{P}_L(t_{\kappa_m})} + 1 \right) \leq \\ &\leq \frac{1}{\kappa_m} \cdot \|f\|'_{1-\alpha,\kappa_m} \left( \frac{\sum_{k=1}^{L-1} \frac{1}{\Gamma(k(\alpha-\beta)+\alpha)}}{\frac{1}{\Gamma(\alpha-\beta)}} + 1 \right) = \end{aligned}$$

$$= \frac{A_L}{\kappa_m} \cdot \|f\|'_{1-\alpha, \kappa_m}.$$

In the above calculations we used the property of gamma Euler function, namely inequalities

$$\Gamma(k(\alpha - \beta) + \alpha) \geq \Gamma(k(\alpha - \beta) + \alpha - \beta)$$

are valid for  $k \geq L, k \in N$  when  $(\alpha - \beta)(L + 1) \geq \gamma_{\min}$ .

We shall omit the proof of parts (3) and (4) as it is analogous to the calculations presented above in detail. ■

**3. Main results.** In this section we shall apply the norms and metrics constructed in Definition 2.5 to prove the existence and uniqueness of the solution for certain fractional differential equations. We begin by studying two one-term equations in the form of

$$t^\beta D_{0+}^\alpha f(t) = \Psi(t, f(t)) \quad (30)$$

$$t^\beta \cdot {}^c D_{0+}^\alpha f(t) = \Psi(t, f(t)) \quad (31)$$

and we shall show that a unique solution for each of the above equations exists in the respective function space and for arbitrary long interval  $[0, b]$ . Thanks to the composition rules from Property 2.3 we can rewrite (30,31) as the equivalent fractional integral equations ( $c \in R$  arbitrary)

$$f(t) = I_{0+}^\alpha t^{-\beta} \Psi(t, f(t)) + ct^{\alpha-1} \quad (32)$$

$$f(t) = I_{0+}^\alpha t^{-\beta} \Psi(t, f(t)) + c. \quad (33)$$

Finally, denoting the mapping on the right-hand side as  $T_{\phi_0}$  in both cases, we obtain equations (30,31) reformulated as the fixed point conditions on the  $C_{1-\alpha}[0, b]$  and  $C[0, b]$  space respectively. Thus, we can solve equations (30,31) applying the Banach theorem on a fixed point, provided the constructed mapping is a contraction on the corresponding function space. The following propositions and corollaries describe the solution for the considered equations.

**PROPOSITION 3.1.** *Let  $\alpha \in (0, 1)$  and  $\alpha - \beta > 0$ . If function  $\Psi \in C([0, b] \times R)$  fulfills the Lipschitz condition*

$$|\Psi(t, x) - \Psi(t, y)| \leq M \cdot |x - y| \quad \forall t \in [0, b] \quad \forall x, y \in R, \quad (34)$$

*then each stationary function  $\phi_0$  of the  $D_{0+}^\alpha$  operator generates unique solution  $f \in C_{1-\alpha}[0, b]$  of equation*

$$t^\beta D_{0+}^\alpha f(t) = \Psi(t, f(t)).$$

*Solution  $f$  is the limit of iterations of mapping  $T_{\phi_0}$ :*

$$f(t) = \lim_{k \rightarrow \infty} (T_{\phi_0})^k \psi(t),$$

*where function  $\psi \in C_{1-\alpha}[0, b]$  is arbitrary.*

*Proof.* Let functions  $g, h \in C_{1-\alpha}[0, b]$  be arbitrary and  $\beta \leq 0$ . Their images  $T_{\phi_0}g$  and  $T_{\phi_0}h$  look as follows

$$T_{\phi_0}g(t) = I_{0+}^\alpha t^{-\beta} \Psi(t, g(t)) + \phi_0(t)$$

$$T_{\phi_0}h(t) = I_{0+}^\alpha t^{-\beta} \Psi(t, h(t)) + \phi_0(t).$$

We estimate the  $d'_{\kappa}$  distance of the above images

$$\begin{aligned} d'_{\kappa}(T_{\phi_0}g, T_{\phi_0}h) &= \| I_{0+}^{\alpha} t^{-\beta} [\Psi(t, g(t)) - \Psi(t, h(t))] \|_{1-\alpha, \kappa} \leq \\ &\leq \| I_{0+}^{\alpha} t^{-\beta} M |g(t) - h(t)| \|_{1-\alpha, \kappa} \leq \\ &\leq \frac{Mb^{-\beta}}{\kappa} \cdot \|g - h\|_{1-\alpha, \kappa} = \frac{Mb^{-\beta}}{\kappa} d'_{\kappa}(g, h). \end{aligned}$$

Thus mapping  $T_{\phi_0}$  is a contraction in space  $\langle C_{1-\alpha}[0, b], d'_{\kappa} \rangle$ , provided  $\kappa$  is large enough. Then, from the Banach theorem it follows that unique fixed point  $f \in C_{1-\alpha}[0, b]$  exists

$$f(t) = T_{\phi_0}f(t).$$

Function  $f$  is the solution of equation (30) and can be explicitly calculated as a limit of iterations of mapping  $T_{\phi_0}$  acting on arbitrary starting function  $\psi$ .

Next, we consider case  $\beta > 0, \alpha - \beta > 0$  and mapping  $T_{\phi_0}$  acting on space  $C_{1-\alpha}[0, b]$  endowed with the  $d''_{\kappa_m}$  metric. Let us note that for any function  $g \in C_{1-\alpha}[0, b]$  its image  $T_{\phi_0}g \in C_{1-\alpha}[0, b]$ . To prove this fact we apply the norm (22) equivalent to the standard norm (2):

$$\begin{aligned} \|T_{\phi_0}g\|'_{1-\alpha, \kappa_m} &\leq \| I_{0+}^{\alpha} t^{-\beta} \Psi(t, g(t)) \|'_{1-\alpha, \kappa_m} + \|\phi_0\|'_{1-\alpha, \kappa_m} \leq \\ &\leq \frac{A_L}{\kappa_m} \| \Psi(t, g(t)) \|'_{1-\alpha, \kappa_m} + \|\phi_0\|'_{1-\alpha, \kappa_m} < \infty. \end{aligned}$$

In the calculations we applied the fact (yielded by the Lipchitz condition) that composed function  $\Psi(t, g(t)) \in C_{1-\alpha}[0, b]$ .

Now, on space  $C_{1-\alpha}[0, b]$ , the  $d''_{\kappa_m}$  distance of images  $T_{\phi_0}g$  and  $T_{\phi_0}h$  looks as follows

$$\begin{aligned} d''_{\kappa_m}(T_{\phi_0}g, T_{\phi_0}h) &= \| I_{0+}^{\alpha} t^{-\beta} [\Psi(t, g(t)) - \Psi(t, h(t))] \|'_{1-\alpha, \kappa_m} \leq \\ &\leq \| I_{0+}^{\alpha} t^{-\beta} M |g(t) - h(t)| \|'_{1-\alpha, \kappa_m} \leq \\ &\leq \frac{M \cdot A_L}{\kappa_m} \cdot \|g - h\|'_{1-\alpha, \kappa_m} = \frac{M \cdot A_L}{\kappa_m} d''_{\kappa_m}(g, h). \end{aligned}$$

Similar to the previous case, we conclude that for  $\kappa_m$  large enough mapping  $T_{\phi_0}$  is a contraction on the  $\langle C_{1-\alpha}[0, b], d''_{\kappa_m} \rangle$  space. Hence fixed point  $f$  exists thanks to the Banach theorem and can be constructed in a way described in the above proposition. ■

**COROLLARY 3.2.** *If assumptions of Proposition 3.1 are fulfilled, then equation*

$$t^{\beta} D_{0+}^{\alpha} f(t) = \Psi(t, f(t))$$

*has unique solution  $f \in C_{1-\alpha}[0, b]$  fulfilling the initial condition*

$$I_{0+}^{1-\alpha} f(0) = d.$$

*Solution  $f$  is the limit of iterations of mapping  $T_{\phi_0}$  generated by stationary function  $\phi_0 = dt^{\alpha-1}/\Gamma(\alpha)$ .*

*Proof.* Due to the one-to-one correspondence between the stationary function and the unique generated solution of equation (30), it is enough to show that function

$\phi_0 = dt^{\alpha-1}/\Gamma(\alpha)$  generates solution  $f$  obeying initial condition  $I_{0+}^{1-\alpha}f(0) = d$ . Solution  $f$  fulfills equation (30) and the following equivalent integral equation

$$f(t) = I_{0+}^{\alpha} t^{-\beta} \Psi(t, f(t)) + \frac{dt^{\alpha-1}}{\Gamma(\alpha)}.$$

Integrating the above equation we arrive at the relation

$$\begin{aligned} I_{0+}^{1-\alpha} f(t) &= I_{0+}^{1-\alpha} I_{0+}^{\alpha} t^{-\beta} \Psi(t, f(t)) + I_{0+}^{1-\alpha} \frac{dt^{\alpha-1}}{\Gamma(\alpha)} = \\ &= I_{0+}^1 t^{(-\beta+1)-1} \Psi(t, f(t)) + d, \end{aligned}$$

where we applied the integration formula [7]:  $I_{0+}^{\gamma} t^{\delta-1} = t^{\delta+\gamma-1} \Gamma(\delta)/\Gamma(\gamma+\delta)$  valid for  $Re(\delta) > 0$  and inequality  $-\beta+1 > 1-\alpha > 0$  in case  $\beta > 0$ .

Now for  $t = 0$  we obtain

$$I_{0+}^{1-\alpha} f(0) = d$$

what ends the proof. ■

Similar results are valid for one-term equation with Caputo derivative (31). We quote Proposition 3.3 and Corollary 3.4 without proof. They both are a straightforward corollary of Properties 2.3 and 2.6 as well as of Proposition 2.7 (part (3) and (4)).

**PROPOSITION 3.3.** *Let  $\alpha \in (0, 1)$  and  $\alpha - \beta > 0$ . If function  $\Psi \in C([0, b] \times R)$  fulfills the Lipschitz condition (34), then each stationary function  $\phi_0$  of the  ${}^c D_{0+}^{\alpha}$  operator generates unique solution  $f \in C[0, b]$  of equation*

$$t^{\beta} \cdot {}^c D_{0+}^{\alpha} f(t) = \Psi(t, f(t)).$$

*Solution  $f$  is the limit of iterations of mapping  $T_{\phi_0}$ :*

$$f(t) = \lim_{k \rightarrow \infty} (T_{\phi_0})^k \psi(t),$$

*where function  $\psi \in C[0, b]$  is arbitrary.*

**COROLLARY 3.4.** *If assumptions of Proposition 3.3 are fulfilled, then equation*

$$t^{\beta} \cdot {}^c D_{0+}^{\alpha} f(t) = \Psi(t, f(t))$$

*has unique solution  $f \in C[0, b]$  fulfilling the initial condition*

$$f(0) = d.$$

*Solution  $f$  is the limit of iterations of mapping  $T_{\phi_0}$  generated by stationary function  $\phi_0 = d$ .*

The results given in the above propositions and corollaries, valid for one-term fractional differential equations can be extended to multi-term equations. In the present paper we shall discuss such equations with basic Riemann-Liouville derivative of order  $\alpha \in (0, 1)$ . Similar procedure can be applied to the analogous equations with Caputo derivative and will be described in a subsequent paper.

We assume that nonlinear part of the fractional differential equation does not depend on

derivative and consider in finite interval a class of equations in the form of:

$$\left[ (t^\beta D_{0+}^\alpha)^m - \sum_{j=1}^{m-1} c_j (t^\beta D_{0+}^\alpha)^j \right] f(t) = \Psi(t, f(t)), \quad (35)$$

where  $\alpha \in (0, 1)$ , coefficients  $c_j \in R$   $j = 1, \dots, m-1$ , function  $\Psi \in C([0, b] \times R)$  and  $\alpha - \beta > 0$ .

Using the composition rules from Property 2.3 we reformulate the above equation as follows

$$(t^\beta D_{0+}^\alpha)^m \left[ \left( 1 - \sum_{j=1}^{m-1} c_j (I_{0+}^\alpha t^{-\beta})^{m-j} \right) f(t) - (I_{0+}^\alpha t^{-\beta})^m \Psi(t, f(t)) \right] = 0. \quad (36)$$

Thus, we note that on the  $C_{1-\alpha}[0, b]$  space equation (35) is equivalent to the following fractional integral equation

$$\left( 1 - \sum_{j=1}^{m-1} c_j (I_{0+}^\alpha t^{-\beta})^{m-j} \right) f(t) - (I_{0+}^\alpha t^{-\beta})^m \Psi(t, f(t)) = \phi_0(t), \quad (37)$$

where function  $\phi_0 \in C_{1-\alpha}[0, b]$  belongs to the kernel of fractional operator  $(t^\beta D_{0+}^\alpha)^m$ :

$$(t^\beta D_{0+}^\alpha)^m \phi_0(t) = 0$$

and is given as ( $\bar{d}_j \in R$   $j = 0, \dots, m-1$ )

$$\phi_0(t) = \sum_{j=0}^{m-1} \frac{\bar{d}_j}{\Gamma((\alpha - \beta)j + \alpha)} t^{(\alpha - \beta)j + \alpha - 1}.$$

Now, we are able to rewrite equations (35,37) as the fixed point condition:

$$f(t) = T_{\phi_0} f(t), \quad (38)$$

where we denoted

$$T_{\phi_0} g(t) := T_m g(t) + \phi_0(t). \quad (39)$$

The proposition below describes the properties of the integral part of the above mapping on the space of continuous weighed functions.

LEMMA 3.5. *Let  $\alpha \in (0, 1)$  and  $\alpha - \beta > 0$ . Mapping  $T_m$  defined as follows for  $g \in C_{1-\alpha}[0, b]$*

$$T_m g(t) := \sum_{j=1}^{m-1} c_j (I_{0+}^\alpha t^{-\beta})^{m-j} g(t) + (I_{0+}^\alpha t^{-\beta})^m \Psi(t, g(t)) \quad (40)$$

*is a contraction in the  $C_{1-\alpha}[0, b]$  space endowed with metric  $d'_\kappa$  for case  $\beta \leq 0$  or  $d''_{\kappa_m}$  for  $\beta > 0$  respectively, provided parameters  $\kappa, \kappa_m$  are large enough and  $\Psi \in C([0, b] \times R)$  fulfills the Lipschitz condition (34).*

The proof of the above lemma is a straightforward result of application of parts (1) or (2) from Proposition 2.7, similar to the proof of Proposition 3.1 presented in detail. Now, we are ready to describe the unique solution of equation (35) in space  $C_{1-\alpha}[0, b]$ .

**THEOREM 3.6.** *Let  $\alpha \in (0, 1)$ ,  $\alpha - \beta > 0$  and  $\Psi \in C([0, b] \times R)$  fulfill the Lipschitz condition (34). Then fractional differential equation*

$$\left[ (t^\beta D_{0+}^\alpha)^m - \sum_{j=1}^{m-1} c_j (t^\beta D_{0+}^\alpha)^j \right] f(t) = \Psi(t, f(t)) \quad (41)$$

has unique solution  $f$  in the  $C_{1-\alpha}[0, b]$  space fulfilling the set of initial conditions:

$$I_{0+}^{1-\alpha} (t^\beta D_{0+}^\alpha)^j f(t) |_{t=0} = d_j, \quad (42)$$

where  $j = 0, \dots, m-1$ . This solution is a limit of the iterations of mapping  $T_{\phi_0}$  (39,40) generated by the stationary function in the form of

$$\phi_0(t) = \sum_{j=0}^{m-1} \frac{\bar{d}_j}{\Gamma((\alpha - \beta)j + \alpha)} t^{(\alpha - \beta)j + \alpha - 1} \quad (43)$$

with coefficients  $\bar{d}_j$  given as follows

$$\bar{d}_0 = d_0 \quad (44)$$

$$\bar{d}_j = \left( d_j - \sum_{l=0}^{j-1} c_{l+m-j} d_l \right) \prod_{k=1}^j \frac{\Gamma((\alpha - \beta)k)}{\Gamma((\alpha - \beta)(k-1) + \alpha)}. \quad (45)$$

*Proof.* Each stationary function  $\phi_0 \in C_{1-\alpha}[0, b]$  of the  $(t^\beta D_{0+}^\alpha)^m$  operator generates a unique solution of equation (35). This follows from Lemma 3.5 which implies that mapping  $T_{\phi_0}$  is a contraction on the  $C_{1-\alpha}[0, b]$  space endowed with the corresponding metric equivalent to (2). To end the proof we shall explicitly show the connection between initial conditions (42) and stationary function (43). Function  $f$  solves simultaneously equation (35) and its integral version

$$f(t) = \sum_{j=1}^{m-1} c_j (I_{0+}^\alpha t^{-\beta})^{m-j} f(t) + (I_{0+}^\alpha t^{-\beta})^m \Psi(t, f(t)) + \sum_{j=0}^{m-1} \frac{\bar{d}_j}{\Gamma((\alpha - \beta)j + \alpha)} t^{(\alpha - \beta)j + \alpha - 1}.$$

Integrating both sides of the above equality we obtain for  $l = 0, \dots, m-1$

$$\begin{aligned} I_{0+}^{1-\alpha} (t^\beta D_{0+}^\alpha)^l f(t) &= \\ &= \sum_{j=1}^{m-1} c_j I_{0+}^{1-\alpha} (t^\beta D_{0+}^\alpha)^l (I_{0+}^\alpha t^{-\beta})^{m-j} f(t) + I_{0+}^{1-\alpha} (t^\beta D_{0+}^\alpha)^l (I_{0+}^\alpha t^{-\beta})^m \Psi(t, f(t)) + \\ &\quad + \sum_{j=0}^{m-1} \frac{\bar{d}_j}{\Gamma((\alpha - \beta)j + \alpha)} I_{0+}^{1-\alpha} (t^\beta D_{0+}^\alpha)^l t^{(\alpha - \beta)j + \alpha - 1}. \end{aligned}$$

Applying the composition rule from Property 2.3 and taking  $t = 0$  we arrive at the following relations

$$d_l = \sum_{j=0}^{l-1} c_{j+m-l} d_j + \bar{d}_l \prod_{k=1}^l \frac{\Gamma((\alpha - \beta)(k-1) + \alpha)}{\Gamma((\alpha - \beta)k)}$$

which yield the explicit form of coefficients  $\bar{d}_l$  (where  $l = 0, \dots, m-1$ ) from formula (44, 45)

$$\bar{d}_0 = d_0$$

$$\bar{d}_j = \left( d_j - \sum_{l=0}^{j-1} c_{l+m-j} d_l \right) \prod_{k=1}^j \frac{\Gamma((\alpha - \beta)k)}{\Gamma((\alpha - \beta)(k - 1) + \alpha)}.$$

The above solution for coefficients  $\bar{d}_j$  is unique and this ends the proof. ■

**4. Final remarks.** In the paper we proved existence-uniqueness results for the one-term and multi-term nonlinear fractional differential equations dependent on the left-sided derivative of given order  $\alpha \in (0, 1)$ . To this aim we extended Bielecki method of equivalent norms [2] known from differential equations theory to fractional differential equations by application of the two-parameter Mittag-Leffler functions in construction of new norms (21-24). The developed method of proving the existence of the solutions yields in arbitrary long interval  $[0, b]$  the unique solution, fulfilling the corresponding set of initial conditions. Let us observe that the standard scaling (using a composed exponential function) is not effective in the considered case as intermediate fractional integral equations (32,33,37) have singular kernels. The new method of construction of the equivalent norms appears to be useful in solving linear multi-term fractional differential equations with basic order  $\alpha \in (n - 1, n)$  and nonlinear multi-term fractional differential equations with Hadamard derivative [11, 12]. The discussed results can be easily extended to the equations given on space of vector functions. Another interesting field, where such method of proof should be effective are sequential fractional differential equations in a sense given in [17, 18], where no basic order of derivatives is assumed.

## References

- [1] D. Baleanu, O.G. Mustafa, *On the global existence of solutions to a class of fractional differential equations*, Comput. Math. Appl. 59 (2010), 1835–1841.
- [2] A. Bielecki, *Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinaires*, Bull. Pol. Acad. Sci. CL.III Vol. IV (1956), 261–264.
- [3] Z.F.A. El-Raheem, *Modification of the application of a contraction mapping method on a class of fractional differential equations*, Appl. Math. Comput. 137 (2003), 371-374.
- [4] M. Khan, S. Hyder Ali, C. Fetecau, Haitao Qi, *Decay of potential vortex for a viscoelastic fluid with fractional Maxwell model*, Appl. Math. Model. 33 (2009), 2526–2533.
- [5] A.A. Kilbas, J.J. Trujillo, *Differential equations of fractional order: methods, results and problems. I*, Appl. Anal. 78 (2001) 153–192.
- [6] A.A. Kilbas, J.J. Trujillo, *Differential equations of fractional order: methods, results and problems. II*, Appl. Anal. 81 (2002) 435–493.
- [7] A.A. Kilbas, H.M. Srivastawa, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
- [8] A.A. Kilbas, M. Rivero, L. Rodríguez-Germá, J.J. Trujillo,  *$\alpha$ -Analytic solutions of some linear fractional differential equations with variable coefficients*, Appl. Math. Comput. 187 (2007) 239–249.
- [9] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longmann-Wiley, New York, 1994.

- [10] M. Klimek, *On Solutions of Linear Fractional Differential Equations of a Variational Type*, The Publishing Office of the Czestochowa University of Technology, Czestochowa, 2009.
- [11] M. Klimek, D. Dziembowski, *On contraction method applied to solve class of sequential fractional differential equations*, in: Proceedings of FDA'10. The 4th Workshop on Fractional Differentiation and Its Applications, Badajoz 2010. I. Podlubny, B.M. Vinagre Jara, YQ. Chen, V.F. Battle, I. Tejado Balsera (eds.). CD-ROM.
- [12] M. Klimek, *Sequential fractional differential equations with Hadamard derivative*, submitted to Commun. Nonlinear Sci. Numer. Simul. .
- [13] V. Lakshmikantham, J. Vasundhara Devi, *Theory of fractional differential equations in a Banach space*, Eur. J. Pure Appl. Math. 1 (2008), 38–45.
- [14] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cambridge, 2009.
- [15] R. Magin, *Fractional Calculus in Bioengineering*, Begell House Publisher, Redding, 2006.
- [16] M.W. Michalski, *Derivatives of noninteger order and their applications*, Dissertationes Math. 328 (1993) 1–47.
- [17] K.S. Miller, B. Ross, *An Introduction to the fractional Calculus and Fractional Differential Equations*, Wiley and Sons, New York, 1993.
- [18] I. Podlubny, *Frcational Differential Equations*, Academic Press, San Diego, 1999.
- [19] M. Rivero, L. Rodríguez-Germá, J.J. Trujillo, *Linear fractional differential equations with variable coefficients*, Appl. Math. Lett. 21 (2008), 892–897.
- [20] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Drivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.
- [21] L. Shan, D. Tong, L. Xue, *Unsteady flow of non-Newtonian visco-elastic fluid in dual porosity media with the fractional derivative*, J. Hydrodyn. B21 (2009), 705–713.
- [22] J. Tian, D. Tong, *The flow analysis of fluids in fractal reservoir with the fractional derivative*, J. Hydrodyn. 18 (2006), 287–293.
- [23] Z.H. Wang, X. Wang, *General solution of the Bagley-Torvik equations with fractional order derivative*, Commun. Nonlinear Sci. Numer. Simul. 15 (2010), 1279–1285.
- [24] B.J. West, M. Bologna, P. Grigolini, *Physics of Fractional Operators*, Springer-Verlag, Berlin, 2003.
- [25] Zhongli Wei, Qingdong Li, Junling Che, *Initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative*, J. Math. Anal. Appl. 367 (2010), 260–272.