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NORM CONVERGENCE OF FEJÉR MEANS OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

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Abstract. The main aim of this paper is to prove that there exist a martingale $f \in H_{1/2}$ such that the Fejér means of the two-dimensional Walsh-Fourier series of the martingale f is not uniformly bounded in the space weak-$L_{1/2}$.

1. Introduction. The first result with respect to the a.e. convergence of the Walsh-Fejér means $\sigma_n f$ is due to Fine [1]. Later, Schipp [5] showed that the maximal operator $\sigma^* f := \sup\limits_n |\sigma_n f|$ is of weak type (1,1), from which the a. e. convergence follows by standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$ but Fujii [2] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1. Fujii's theorem was extended by Weisz [8]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh-Fourier series is bounded from the martingale Hardy space $H_p(G)$ to the space $L_p(G)$ for $p > 1/2$. Simon [6] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In the endpoint case $p = 1/2$ Weisz [11] proved that σ^* is bounded from the Hardy space $H_{1/2}(G)$ to the space weak-$L_{1/2}(G)$. In [3] the author proved that the maximal operator σ^* is not bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$. By interpolation it follows that σ^* is not bounded from the Hardy space H_p to the space weak-L_p for any $0 < p < 1/2$.

For the two-dimensional Walsh-Fourier series Weisz [9, 10] proved that the following is true

THEOREM W1. Let $p > 1/2$. Then the maximal operator σ^* is bounded from the Hardy space H_p to the space L_p.

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The author [4] proved that in Theorem W1, for the maximal operator σ^*, the assumption $p > 1/2$ is essential. Moreover, we prove, that the following is true.

THEOREM G. The maximal operator σ^* is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

Weisz [9, 10] considered the norm convergence of Fejér means of the two-dimensional Walsh-Fourier series. In particular, the following is true

THEOREM W2. Let $p > 1/2$. Then

$$\|\sigma_{n,m} f\|_{H_p} \leq c_p \|f\|_{H_p} \left(f \in H_p\right).$$

In [9] Weisz conjectured that for the uniformly boundedness of the operator $\sigma_{n,m}$ from the Hardy space $H_p\left(G \times G\right)$ to the space $H_p\left(G \times G\right)$ the assumption $p > 1/2$ is essential. We give answer to the question, moreover, we prove that the operator $\sigma_{n,n}$ is not uniformly bounded from the Hardy space $H_{1/2}\left(G \times G\right)$ to the space weak$-L_{1/2}\left(G \times G\right)$. In particular, the following is true.

THEOREM 1.1. *There exist a martingale $f \in H_{1/2}\left(G \times G\right)$ such that*

$$\sup_n \|\sigma_{n,n} f\|_{weak-L_{1/2}} = +\infty.$$

2. Dyadic Hardy spaces. Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0, 1\}\left(k \in \mathbf{N}\right)$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0\left(x\right) := G, \;\; I_n\left(x\right) := \; I_n\left(x_0, ..., x_{n-1}\right) := \left\{y \in G : \; y = \left(x_0, ..., x_{n-1}, y_n, y_{n+1}, ...\right)\right\},$$

$$\left(x \in G, n \in \mathbf{N}\right).$$

These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G, $\; I_n := I_n\left(0\right) \; \left(n \in \mathbf{N}\right)$. Set $e_n := (0, ..., 0, 1, 0, ...) \in G$ the n th coordinate of which is 1 and the rest are zeros $\left(n \in \mathbf{N}\right)$.

For $k \in \mathbf{N}$ and $x \in G$ denote

$$r_k\left(x\right) := \left(-1\right)^{x_k}$$

the k-th Rademacher function.

The dyadic rectangles are of the form

$$I_{n,m}\left(x, y\right) := I_n\left(x\right) \times I_m\left(y\right).$$

The σ-algebra generated by the dyadic rectangles $\left\{I_{n,m}\left(x, y\right) : \left(x, y\right) \in G \times G\right\}$is denoted by $F_{n,m}$.

The norm (or quasinorm) of the space $L_p(G\times G)$ is defined by

$$\|f\|_p := \left(\int\limits_{G\times G} |f(x,y)|^p \, d\mu(x,y)\right)^{1/p} \qquad (0<p<+\infty).$$

The space weak-$L_p(G\times G)$ consists of all measurable functions f for which

$$\|f\|_{weak-L_p(G\times G)} := \sup_{\lambda>0} \lambda\mu\left(|f|>\lambda\right)^{1/p} < +\infty.$$

Denote by $f=\left(f^{(n,m)}, n,m\in N\right)$ two parameter martingale with respect to $(F_{n,m}, n,m\in \mathbf{N})$ (for details see, e. g. [7, 10]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n,m\in N}\left|f^{(n,m)}\right|.$$

In case $f\in L_1(G\times G)$, the maximal function can also be given by

$$f^*(x,y) = \sup_{n,m\in\mathbf{N}} \frac{1}{\mu\left(I_{n,m}(x,y)\right)}\left|\int\limits_{I_{n,m}(x,y)} f(u,v)\, d\mu(u,v)\right|,$$

$$(x,y)\in G\times G,$$

For $0<p<\infty$ the Hardy martingale space $H_p(G\times G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

3. Walsh system and Fejér means. Let $n\in\mathbf{N}$, then $n=\sum\limits_{i=0}^{\infty} n_i 2^i$, where $n_i\in\{0,1\}$ $(i\in\mathbf{N})$, i. e. n is expressed in the number system of base 2. Denote $|n| := \max\{j\in\mathbf{N} : n_j\neq 0\}$, that is, $2^{|n|}\leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty}\left(r_k(x)\right)^{n_k} = r_{|n|}(x)\,(-1)^{\sum\limits_{k=0}^{|n|-1} n_k x_k} \qquad (x\in G, n\in\mathbf{P}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, \text{if } x\in I_n, \\ 0, \ \ \text{if } x\in G\backslash I_n. \end{cases} \tag{1}$$

The Fejér kernel of order n of the Walsh-Fourier series is defined by

$$K_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(x).$$

The rectangular partial sums of the double Walsh-Fourier series are defined as follows:

$$S_{M,N}f(x,y) := \sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\widehat{f}(i,j)\,w_i(x)\,w_j(y)\,,$$

where the number

$$\widehat{f}(i,j) = \int\limits_{G\times G} f(x,y)\,w_i(x)\,w_j(y)\,d\mu(x,y)$$

is said to be the (i,j)th Walsh-Fourier coefficient of the function f.

If $f \in L_1(G\times G)$ then it is easy to show that the sequence $(S_{2^n,2^m}(f) : n,m \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n,m)} : n,m \in \mathbf{N})$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i,j) = \lim_{\min(k,l)\to\infty}\int\limits_{G\times G} f^{(k,l)}(x,y)\,w_i(x)\,w_j(y)\,d\mu(x,y)\,. \tag{2}$$

The Walsh-Fourier coefficients of $f \in L_1(G\times G)$ are the same as the ones of the martingale $(S_{2^n,2^m}(f) : n,m \in \mathbf{N})$ obtained from f.

For $n,m \in \mathbf{P}$ and a martingale f the Fejér mean of order (n,m) of the double Walsh-Fourier series of the martingale f is given by

$$\sigma_{n,m}f(x,y) = \frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} S_{i,j}f(x,y)\,.$$

For the martingale f the maximal operator is defined by

$$\sigma^* f(x,y) = \sup_{n,m}|\sigma_{n,m}f(x,y)|\,.$$

A function $a \in L_2$ is called a rectangle p-atom if there exists a dyadic rectangle R such that

$$\begin{cases} \mathtt{supp}(a) \subset R, \\ \|a\|_2 \le |R|^{1/2-1/p} \\ \int\limits_G a(x,y)\,d\mu(x) = \int\limits_G a(x,y)\,d\mu(y) = 0 \ \mathtt{for}\ \textit{all}\ \ x,y \in G. \end{cases}$$

The basic result of atomic decomposition is the following one.

Theorem W3. A martingale $f = \left(f^{(n,m)} : n,m \in \mathbf{N}\right)$ is in $H_p\,(0 < p \le 1)$ if there exists a sequence $(a_k, k \in \mathbf{N})$ of rectangle p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that for every $n,m \in \mathbf{N}$,

$$\sum_{k=0}^{\infty}\mu_k S_{2^n,2^m}a_k = f^{(n,m)},$$

$$\sum_{k=0}^{\infty}|\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \le \inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}.$$

4. Auxiliary Result. In order to prove theorem we need the following lemma .

LEMMA 4.1. *[4]Let* $2 < A \in \mathbf{P}$ *and* $q_A := 2^{2A} + 2^{2A-2} + \cdots + 2^2 + 2^0$. *Then*

$$q_{A-1} \left| K_{q_{A-1}} (x) \right| \geq 2^{2m+2s-3}$$

for $x \in I_{2A}^{m,s} := I_{2A} (0, ..., 0, x_{2m} = 1, 0, ..., 0, x_{2s} = 1, x_{2s+1}, ..., x_{2A-1})$, $m = 0, 1, ..., A-3, \quad s = m+2, m+3, ..., A-1.$

5. Proof of main result. *Proof of Theorem 1.1.* Since $2^m/m \uparrow \infty$ it is easy to show that there exists an increasing sequence of positive integers $\{m_k : k \in \mathbf{P}\}$ such that

$$\sum_{k=1}^{\infty} \frac{1}{m_k^{1/2}} < \infty, \tag{3}$$

$$\sum_{l=0}^{k-1} \frac{2^{8m_l}}{m_l} < \frac{2^{8m_k}}{m_k}, \tag{4}$$

$$\frac{2^{8m_{k-1}}}{m_{k-1}} < \frac{2^{m_k}}{km_k}. \tag{5}$$

Let

$$f^{(A,B)} (x, y) := \sum_{\{l: 2m_l < \min(A,B)\}} \lambda_l a_l (x, y) .$$

where

$$\lambda_l := \frac{1}{m_l}$$

and

$$a_l (x, y) := 2^{4m_l} \left(D_{2^{2m_l+1}} (x) - D_{2^{2m_l}} (x) \right) \left(D_{2^{2m_l+1}} (y) - D_{2^{2m_l}} (y) \right) .$$

First, we prove that the martingale $f := \left(f^{(A,B)} : A, B \in \mathbf{N} \right)$ belongs to the Hardy space $H_{1/2} (G \times G)$. Indeed, since

$$\| a_l \|_2 \leq c 2^{6m_l},$$

$$S_{2^A, 2^B} a_k (x, y) = \begin{cases} 0, & \texttt{if} \ \min (A, B) \leq 2m_k, \\ a_k (x, y) , & \texttt{if} \min (A, B) > 2m_k, \end{cases}$$

we can write

$$f^{(A,B)} (x, y) := \sum_{\{l: 2m_l < \min(A,B)\}} \lambda_l a_l (x, y) = \sum_{k=0}^{\infty} \lambda_k S_{2^A, 2^B} a_k (x, y)$$

from (3) and Theorem W3 we conclude that $f \in H_{1/2} (G \times G)$.

Now, we investigate the Fourier coefficients. Since

$$\int_{G \times G} f^{(A,B)} (x, y) \, w_i (x) \, w_j (y) \, d\mu (x, y)$$

$$= \begin{cases} 0, & (i,j) \notin \bigcup\limits_{k=0}^{\infty} \{2^{2m_k},...,2^{2m_k+1}-1\} \\ & \times\{2^{2m_k},...,2^{2m_k+1}-1\}, \\ 0, & (i,j) \in \{2^{2m_k},...,2^{2m_k+1}-1\} \\ & \times\{2^{2m_k},...,2^{2m_k+1}-1\}, \min(A,B) \le 2m_k, \\ \frac{2^{4m_k}}{m_k}, & (i,j) \in \{2^{2m_k},...,2^{2m_k+1}-1\} \\ & \times\{2^{2m_k},...,2^{2m_k+1}-1\}, \min(A,B) > 2m_k, \end{cases}$$

we can write (see (2))

$$\widehat{f}(i,j) = \begin{cases} \frac{2^{4m_k}}{m_k}, & (i,j) \in \{2^{2m_k},...,2^{2m_k+1}-1\} \\ & \times\{2^{2m_k},...,2^{2m_k+1}-1\}, k=1,2,..., \\ 0, & (i,j) \notin \bigcup\limits_{k=1}^{\infty} \{2^{2m_k},...,2^{2m_k+1}-1\} \\ & \times\{2^{2m_k},...,2^{2m_k+1}-1\}. \end{cases} \tag{6}$$

Let $q_{m_k} := 2^{2m_k} + 2^{2m_k-2} + \cdots + 2^2 + 2^0$. Then we can write

$$\begin{aligned} \sigma_{q_{m_k},q_{m_k}} f(x,y) &= \frac{1}{q_{m_k}^2} \sum_{i=0}^{q_{m_k}-1} \sum_{j=0}^{q_{m_k}-1} S_{i,j} f(x,y) \qquad (7) \\ &= \frac{1}{q_{m_k}^2} \sum_{i=0}^{2^{2m_k}-1} \sum_{j=0}^{2^{2m_k}-1} S_{i,j} f(x,y) \\ &\quad + \frac{1}{q_{m_k}^2} \sum_{i=2^{2m_k}}^{q_{m_k}-1} \sum_{j=0}^{2^{2m_k}-1} S_{i,j} f(x,y) \\ &\quad + \frac{1}{q_{m_k}^2} \sum_{i=0}^{2^{2m_k}-1} \sum_{j=2^{2m_k}}^{q_{m_k}-1} S_{i,j} f(x,y) \\ &\quad + \frac{1}{q_{m_k}^2} \sum_{i=2^{2m_k}}^{q_{m_k}-1} \sum_{j=2^{2m_k}}^{q_{m_k}-1} S_{i,j} f(x,y) \\ &= I + II + III + IV. \end{aligned}$$

Let $(i,j) \in \left\{2^{2m_k},...,q_{m_k}-1\right\} \times \left\{2^{2m_k},...,q_{m_k}-1\right\}$. Then from (6) we have

$$\begin{aligned} S_{i,j} f(x,y) &= \sum_{v=0}^{i-1} \sum_{\mu=0}^{j-1} \widehat{f}(\nu,\mu) w_\nu(x) w_\mu(y) \\ &= \sum_{l=1}^{k-1} \sum_{\nu=2^{m_l}}^{2^{m_l+1}-1} \sum_{\mu=2^{m_l}}^{2^{m_l+1}-1} \widehat{f}(\nu,\mu) w_\nu(x) w_\mu(y) \\ &\quad + \sum_{\nu=2^{2m_k}}^{i-1} \sum_{\mu=2^{2m_k}}^{j-1} \widehat{f}(\nu,\mu) w_\nu(x) w_\mu(y) \end{aligned}$$

$$= \sum_{l=1}^{k-1} \frac{2^{4m_l}}{m_l} \left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) \times \left(D_{2^{2m_l+1}}(y) - D_{2^{2m_l}}(y)\right)$$

$$+\frac{2^{4m_k}}{m_k}\left(D_i(x)-D_{2^{2m_k}}(x)\right)\left(D_j(y)-D_{2^{2m_k}}(y)\right). \tag{8}$$

substitute (8) in IV, we have

$$\begin{aligned} & IV \qquad (9)\\ = & \frac{1}{q_{m_k}^2}\left(q_{m_k}-2^{2m_k}\right)^2\sum_{l=1}^{k-1}\frac{2^{4m_l}}{m_l}\left(D_{2^{2m_l+1}}(x)-D_{2^{2m_l}}(x)\right)\\ & \times\left(D_{2^{2m_l+1}}(y)-D_{2^{2m_l}}(y)\right)\\ & +\frac{1}{q_{m_k}^2}\frac{2^{4m_k}}{m_k}\sum_{i=2^{2m_k}}^{q_{m_k}-1}\sum_{j=2^{2m_k}}^{q_{m_k}-1}\left(D_i(x)-D_{2^{2m_k}}(x)\right)\\ & \times\left(D_j(y)-D_{2^{2m_k}}(y)\right)\\ = & IV_1+IV_2. \end{aligned}$$

Since

$$D_{j+2^{2m_k}}(x)=D_{2^{2m_k}}(x)+w_{2^{2m_k}}(x)D_j(x), \qquad j=0,1,...,2^{2m_k}-1.$$

for IV_2 we can write

$$\begin{aligned} IV_2 &= \frac{1}{q_{m_k}^2}\frac{2^{4m_k}}{m_k}w_{2^{2m_k}}(x)w_{2^{2m_k}}(y)\sum_{i=0}^{q_{m_k-1}-1}D_i(x)\sum_{j=0}^{q_{m_k-1}-1}D_j(y)\\ &= \frac{1}{q_{m_k}^2}\frac{2^{4m_k}}{m_k}w_{2^{2m_k}}(x)w_{2^{2m_k}}(y)q_{m_k-1}^2K_{q_{m_k-1}}(x)K_{q_{m_k-1}}(y). \end{aligned} \tag{10}$$

Since

$$\left|D_{2^n}(x)\right|\le 2^n, \quad n\in N, \quad x\in G$$

by (4) and (5) we obtain

$$|IV_1|\le C\sum_{l=1}^{k-1}\frac{2^{8m_l}}{m_l}\le C\frac{2^{m_k}}{km_k}. \tag{11}$$

Combining (9)-(11) we have

$$IV\ge\frac{Cq_{m_k-1}^2}{m_k}\left|K_{q_{m_k-1}}(x)\right|\left|K_{q_{m_k-1}}(y)\right|-\frac{C2^{m_k}}{km_k}. \tag{12}$$

Let

$$\begin{aligned} (i,j)\in & \left(\left\{2^{2m_k},...,q_{m_k}-1\right\}\times\left\{0,1,...,2^{2m_k}-1\right\}\right)\\ & \cup\left(\left\{0,1,...,2^{2m_k}-1\right\}\times\left\{2^{2m_k},...,q_{m_k}-1\right\}\right)\\ & \cup\left(\left\{0,1,...,2^{2m_k}-1\right\}\times\left\{0,1,...,2^{2m_k}-1\right\}\right). \end{aligned}$$

Then from (6), (4) and (5) it is easy to show that

$$\begin{aligned} \left|S_{i,j}f(x,y)\right| &\le \sum_{l=0}^{k-1}\sum_{\nu=2^{2m_l}}^{2^{2m_l+1}-1}\sum_{\mu=2^{2m_l}}^{2^{2m_l+1}-1}\left|\widehat{f}(\nu,\mu)\right|\\ &\le \sum_{l=0}^{k-1}\frac{2^{8m_l}}{m_l}\le\frac{C2^{m_k}}{km_k}. \end{aligned}$$

Consequently

$$|I| \leq \frac{1}{q_{m_k}^2} \sum_{i=0}^{2^{2m_k}-1} \sum_{j=0}^{2^{2m_k}-1} |S_{i,j} f(x,y)|$$

$$\leq C \frac{2^{4m_k}}{q_{m_k}^2} \frac{2^{m_k}}{km_k} \leq \frac{C2^{m_k}}{km_k} \tag{13}$$

$$|II| \leq \frac{2^{2m_k}(q_{m_k} - 2^{2m_k})}{q_{m_k}^2} \frac{2^{m_k}}{km_k} \leq C \frac{2^{m_k}}{km_k}, \tag{14}$$

$$|III| \leq \frac{C2^{m_k}}{km_k}. \tag{15}$$

Combining (7), (9)-(15) we obtain that

$$\left|\sigma_{q_{m_k},q_{m_k}} f(x,y)\right| \geq \frac{Cq_{m_k-1}^2}{m_k} |K_{q_{m_k-1}}(x)||K_{q_{m_k-1}}(y)| - \frac{C2^{m_k}}{km_k}. \tag{16}$$

Let $(x,y) \in I_{2m_k}^{l_1,l_1+2} \times I_{2m_k}^{l_2,l_2+2}$, $(l_1,l_2) \in \{0,1,...,m_k-3\} \times \{0,1,...,m_k-3\}$. Then from Lemma 4.1 we can write

$$q_{m_k-1}\left|K_{q_{m_k-1}}(x)\right| \geq C2^{4l_1}$$

and

$$q_{m_k-1}\left|K_{q_{m_k-1}}(y)\right| \geq C2^{4l_2},$$

consequently,

$$q_{m_k-1}^2\left|K_{q_{m_k-1}}(x)\right|\left|K_{q_{m_k-1}}(y)\right| \geq C2^{4l_1+4l_2},$$

$$\left|\sigma_{q_{m_k},q_{m_k}} f(x,y)\right| \geq \frac{C}{m_k} 2^{4l_1+4l_2} - \frac{C2^{m_k}}{km_k}. \tag{17}$$

Denote

$$A(m_k) := \left\{(l_1,l_2) : 0 \leq l_2 \leq m_k - 3, 0 \leq l_1 \leq \frac{m_k}{4}, l_1 + l_2 \geq \frac{m_k}{4}\right\}$$

and

$$\alpha_k := \frac{C2^{m_k}}{m_k}.$$

Since (see (17) and $(l_1,l_2) \in A(m_k)$)

$$\begin{aligned}\left|\sigma_{q_{m_k},q_{m_k}} f(x,y)\right| &\geq \frac{C}{m_k} 2^{m_k} - \frac{C2^{m_k}}{km_k} \\ &\geq \frac{C2^{m_k}}{m_k} = \alpha_k\end{aligned}$$

we have

$$\mu\left\{(x,y) \in G \times G : \left|\sigma_{q_{m_k},q_{m_k}} f(x,y)\right| \geq C\alpha_k\right\}$$

$$\geq \sum_{(l_1,l_2)\in A(m_k)} \mu\left\{(x,y) \in I_{2m_k}^{l_1,l_1+2} \times I_{2m_k}^{l_2,l_2+2} : \left|\sigma_{q_{m_k},q_{m_k}} f(x,y)\right| \geq \alpha_k\right\}$$

$$\geq C \sum_{l_1=0}^{[m_k/4]} \sum_{l_2=[m_k/4]-l_1}^{m_k-3} \sum_{x_{2l_1+5}=0}^{1} \cdots \sum_{x_{2m_k-1}=0}^{1} \sum_{x_{2l_2+5}=0}^{1} \cdots \sum_{x_{2m_k-1}=0}^{1}$$

$$\mu\left(I_{2m_k}^{l_1,l_1+2} \times I_{2m_k}^{l_2,l_2+2}\right)$$

$$\geq C \sum_{l_1=0}^{[m_k/4]} \sum_{l_2=[m_k/4]-l_1}^{m_k-3} \frac{1}{2^{2l_1+2l_2}} \geq \frac{Cm_k}{2^{m_k/2}}.$$

Consequently

$$\alpha_k \left(\mu\left\{(x,y) : \left|\sigma_{q_{m_k},q_{m_k}} f(x,y)\right| \geq C\alpha_k\right\}\right)^2$$

$$\geq C \frac{2^{m_k}}{m_k} \frac{m_k^2}{2^{m_k}} = Cm_k \to \infty \quad as \quad k \to \infty,$$

$$\sup_k \left\|\sigma_{q_{m_k},q_{m_k}} f\right\|_{weak-L_{1/2}} :=$$

$$= \sup_k \sup_{\lambda>0} \lambda \left(\mu\left\{(x,y) \in G \times G : \sigma_{q_{m_k},q_{m_k}} f(x,y) > \lambda\right\}\right)^2 = +\infty.$$

Theorem 1.1 is proved. ■

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