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SMALL AND BIG PROBABILITY WORLDS

MARC YOR

Laboratoire de Probabilités et Modèles Aléatoires,
Universités Paris VI et VII, 4 Place Jussieu – Case 188,
F-75252 Paris Cedex 05
Institut Universitaire de France

Introduction.

I.1 In many branches of mathematics, it is important to find the "right framework" in which a problem may be phrased and solved. I believe this plays even a more important role in Probability Theory, where a random phenomenon may be considered as part of a more complex one, or on the contrary, one may choose, given a random phenomenon, to study only some of its aspects.

I.2 In this lecture, I would like to give a number of examples where

- either, in the middle of a complex stochastic framework, one may isolate some features involving a few random variables which may serve as a guideline to more sophisticated results,
- or, on the contrary, while looking at some reasonably simple, finite-dimensional stochastic phenomena, one finds some reward in embedding this phenomena in an infinite dimensional stochastic framework.

I.3 I would like to call the first attitude: "Stripping a random phenomenon to its skeleton", and I have often used this reduction to create exercises in first courses of probability theory, starting from much more complex material involving e.g. studies of Brownian functionals. Several such examples are given in Part A of this lecture.

I.4 The second attitude which I would like to call: "Embedding (or Dressing) a finite-dimensional random phenomenon into an infinite dimensional one" is, in fact, borrowed

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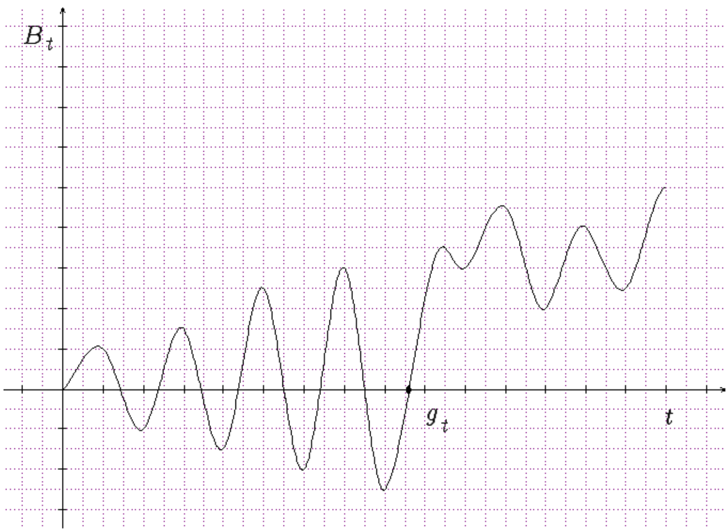
from K. Itô, and may be called "Itô's principle". Indeed, in the Foreword to his Selected Papers volume, Itô writes, in substance: "After a while, it became my habit to consider even finite dimensional probabilistic objects within an infinite dimensional framework". Itô wanted thus to explain how he came to think of the Poisson point process of (say, Brownian) excursions. As is now well recognized and understood, Itô's PPP of excursions allows to obtain myriads of results about "individual excursions"; D. Williams called this "the miracle of excursion theory".

I.5 Let us point out that the two "attitudes" are in fact deeply ingrained in Probability Theory; indeed, conditioning with respect to a σ-field corresponds to focussing attention to some part of a random phenomenon, while freezing another part of the same; on the other hand, one often needs to conveniently enlarge a probability set-up $(\Omega, \mathcal{F}, P)$ into $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, and this for various reasons: the very definition of such enlargements is done very carefully e.g. in Getoor–Sharpe [9] to prepare for their conformal version of the Dambis–Dubins–Schwarz representation of continuous martingales; let us also mention Skorokhod's change of probability space in order to transform convergence in law into convergence almost everywhere, a deep modification (see: Skorokhod [20], Shiryaev [19]) which I always refrain to present to students in Probability right after explaining at length about the different notions of convergence in Probability Theory.

I.6 Throughout this lecture, I allowed myself to use some straight forward abbreviations, which should cause no trouble for the reader, e.g: N and N' denote two reduced independent Gaussian variables, and so forth... I believe this reduces (unnecessary) lenghty precisions.

I.7 Throughout this lecture, the arc sine law, and some of its occurrences with Brownian motion, shall be a recurrent theme:



A (stylized!) Brownian trajectory

Both $\frac{1}{t}g_t \equiv \frac{1}{t}\sup\{s \leq t : B_s = 0\}$, and $\frac{1}{t}A_t^+ \equiv \frac{1}{t}\int_0^t ds 1_{(B_s>0)}$ are arc-sine law distributed, that is:

$$P\Big(\frac{1}{t}g_t \in dx\Big) \equiv P\Big(\frac{1}{t}A_t^{(+)} \in dx\Big) \equiv \frac{dx}{\overline{u}\sqrt{x(1-x)}}(x \in (0,1))$$

In order not to distract the reader by giving too many illustrations of Stripping and Dressing, I shall not discuss the "starred items", whose study is less central than those of the "unstarred ones".

Plan of the lecture:

• ***Warming up:*** A simple family of probability distributions: the beta gamma algebra.

• ***A-Stripping***
→ A random phenomenon to its skeleton.
→ Some (deep) Brownian facts to the beta-gamma algebra.
(A.1) The theorems of Lévy and Pitman about RBM and BES(3).
(A.2) The arc sine law of Paul Lévy.
(A.3) BM up to an independent exponential time.
(A.4) Tsirel'son's equation.

• ***B-Dressing up***
(B.1) Bringing in a BM to prove a result involving 1 or 2 r.v's. Neveu's proof of the hypercontractivity of the OU semigroup.
(B.2) Calling upon a second BM to help a BM: The Ciesielski–Taylor identities.
(B.3) Calling upon the Wiener sheet to help a BM.
(B.4) From the arc sine law to the (infinite dim.) Poisson Dirichlet laws.
(B.5) Behind the asymptotic studies of planar BM add. functionals, there lies a phantom world.
(B.6) Calling upon Itô's PPP of excursions to understand BM.
(B.7) Calling upon Markovian predictors to help understand non-Markovian processes: Knight's prediction theory.

Warming up

• **Warning:** An inequality in law: $f(X,Y) \overset{\text{(law)}}{=} g(U,V)$ is understood with X and Y independent on one hand, and U and V independent on the other hand.
• **Gamma variables:** for $a > 0$, we denote by γ_a a gamma variable with parameter a, i.e:

$$P(\gamma_a \in dx) = \frac{x^{a-1}e^{-x}dx}{\Gamma(a)} \qquad (x > 0)$$

• **Beta variables:** for $a, b > 0$, we denote by $\beta_{a,b}$ a beta variable with parameters a and b, i.e:

$$P(\beta_{a,b} \in dx) = \frac{x^{a-1}(1-x)^{b-1}dx}{B(a,b)} \qquad (0 < x < 1)$$

• **The beta-gamma algebra two main results:**

i) $\gamma_a + \gamma_b \overset{\text{(d)}}{=} \gamma_{a+b}$
ii) $\frac{\gamma_a}{\gamma_a+\gamma_b} \overset{\text{(d)}}{=} \beta_{a,b}$ and is independent from $(\gamma_a + \gamma_b)$

In other terms:

$$(\gamma_a, \gamma_b) \overset{\text{(d)}}{=} (\beta_{a,b}; 1 - \beta_{a,b})\gamma_{a+b}$$

with independence on both sides, i.e: γ_a and γ_b are independent; $\beta_{a,b}$ is independent from γ_{a+b}.

• **Examples and Notation:** We denote by N and N' a couple of independent standard Gaussian r.v.'s.

$*$ $\boxed{\text{a=b=1/2}}$ Note that: $N^2 \overset{\text{(d)}}{=} 2\gamma_{1/2}$

Thus: $\dfrac{N^2}{N^2+N'^2} \overset{\text{(d)}}{=} \beta_{1/2,1/2}$ (an arc sine dist. r.v.).

Box–Müller: $N + iN' \equiv \sqrt{N^2+N'^2}\exp(i\theta) = (\log(1/U))\exp(i\theta)$, with $U \overset{\text{(d)}}{=} \beta_{1,1}$; $\frac{1}{2\pi}\theta \overset{\text{(d)}}{=} \beta'(1,1)$.

$*$ $\boxed{\text{a=b=1}}$ $\mathfrak{e}$ and $\mathfrak{e}$' denote two standard, independent, exponential variables.

$$\mathfrak{e}, \mathfrak{e}' \overset{\text{(d)}}{=} \gamma_1$$
$$\frac{\mathfrak{e}}{\mathfrak{e}+\mathfrak{e}'} \overset{\text{(d)}}{=} U \equiv \beta_{1,1}$$

A-Stripping

(A.1) *The theorems of Lévy and Pitman about RBM and BES(3).* $(B_t, t \geq 0)$1 dim BM; $S_t = \sup_{s\leq t} B_s$; $L_t \equiv \lim_{\varepsilon\to 0} \dfrac{1}{2\varepsilon}\displaystyle\int_0^t ds 1_{(|B_s|\leq\varepsilon)}$

Then:

– (Lévy)

$$(S_t - B_t, S_t; t \geq 0) \overset{\text{(d)}}{=} (|B_t|, L_t; t \geq 0)$$

Moreover, the symmetry principle of D. André states that:

$$\text{for fixed } t, |B_t| \overset{\text{(d)}}{=} S_t \overset{\text{(d)}}{=} L_t \overset{\text{(d)}}{=} \sqrt{t}|N|$$

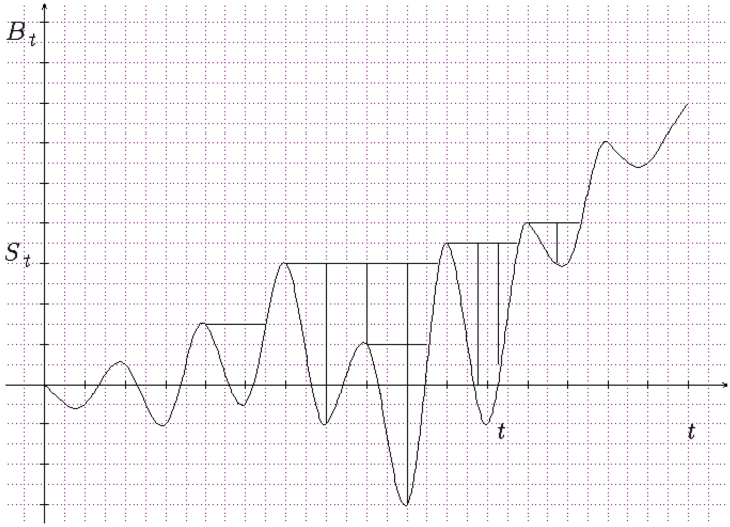
– (Pitman; 1975)

$$(2S_t - B_t, S_t; t \geq 0) \overset{\text{(d)}}{=} (|B_t| + L_t, L_t; t \geq 0) \overset{\text{(d)}}{=} (R_t, \inf_{s\geq t} R_s; t \geq 0)$$

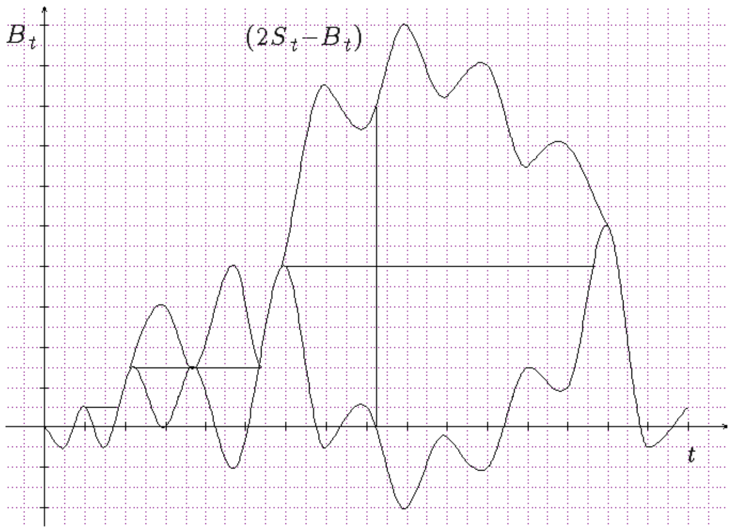
where $(R_t, t \geq 0)$ denotes the 3-dimensional Bessel process, starting from 0.

Corollary 1. *If $\sigma_a = \inf\{t : B_t > a\}$, and $\tau_\ell = \inf\{t : L_t > \ell\}$, then:*

$$\sigma_a \overset{\text{(law)}}{=} a^2/N^2 \,;\; \tau_\ell \overset{\text{(law)}}{=} \ell^2/N^2$$



(the vertical lengths represent: $S_t - B_t$)



Now, from the beta-gamma algebra: Conditionally on $|B_t| + L_t = r$, and even the whole σ-field of $(|B_s| + L_s, s \le t)$, $|B_t|$ (or L_t) is uniform on $[0, r]$, i.e: $|B_t| \overset{(d)}{=} r\mathbf{U}$.
(A.2) *The arc sine law of Paul Lévy.*

$$\Big(\frac{1}{t} g_t \overset{(d)}{=}\Big) \frac{1}{t} A_t^{(+)} \overset{(d)}{=} \beta_{1/2,1/2} \text{ ; why?}$$

In his 1939 paper: Sur certains processus stochastiques homognes, P. Lévy [12] showed that

$$\frac{1}{t}(A_t^{(+)}, A_t^{(-)}, L_t^2) \overset{(d)}{=} \frac{1}{\tau_\ell}(A_{\tau_\ell}^{(+)}, A_{\tau_\ell}^{(-)}, \ell^2) \tag{1}$$

where $\tau_\ell \equiv \inf\{t : L_t > \ell\}$.

This is not at all obvious as at time t, B_t is in $\mathbb{R}_+$ with probability (1/2).

There are other times S called: Admissible times by J. Pitman and I, at which the triplet as in (1), but with t replaced by S, has the same distribution. Now, back to (1):

$$(A^{(+)}_{\tau_\ell}, A^{(-)}_{\tau_\ell}) \overset{(d)}{=} \frac{\ell^2}{4}(T,T') ,$$

so that, from (1), again:

$$\frac{1}{t}A^{(+)}_t \overset{(d)}{=} \frac{T}{T+T'} \overset{(d)}{=} \frac{N^2}{N^2+N'^2}$$

(since: $T \overset{(d)}{=} 1/N^2$, as explained before, in Corollary 0.1, where instead of T we used σ_1).

(A.3) *BM up to an independent exponential time (denoted here by T!).* It is well known that:

$$\Big(\frac{1}{\sqrt{g_t}}B_{ug_t}, u \leq 1\Big) \text{ is a Standard Brownian Bridge,}$$

whereas: $\big(\frac{1}{\sqrt{t-g_t}}B_{g_t+u(t-g_t)}, u \leq 1\big)$ is a Standard Brownian Meander.

Now:

$$\begin{aligned}(g_T, T-g_T) &\overset{(d)}{=} (\beta_{1/2,1/2}, 1-\beta_{1/2,1/2})2\mathfrak{e} \\ &\overset{(d)}{=} (N^2, N'^2)\end{aligned}$$

Applications: a) We consider the principal value:

$$H_t = \int_0^t \frac{ds}{B_s} \equiv \lim_{\varepsilon\to 0}\int_0^t \frac{ds}{B_s}1_{(|B_s|\geq\varepsilon)}$$

and we denote: $H^-_t = H_{g_t}$; $H^+_t = H_t - H_{g_t}$.

Then:

$$H_T = H^+_T + H^-_T$$

(where it is well known that H^+_T and H^-_T are independent) and we find:

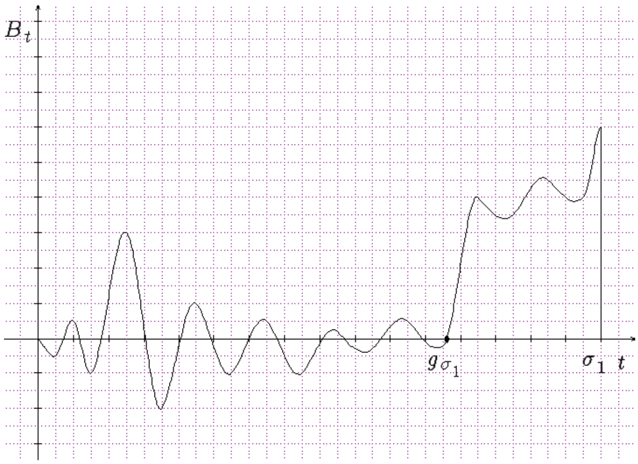
$$E\Big[\exp\Big(i\frac{\lambda}{\pi}H^-_T\Big)\Big] = \frac{\tanh(\lambda)}{\lambda} \; ; \; E\Big[\exp\Big(i\frac{\lambda}{\pi}H^+_T\Big)\Big] = \frac{\lambda}{\sinh(\lambda)}$$

$$E\Big[\exp(i\lambda H_T)\Big] = \frac{1}{\cosh(\lambda)}$$

Thus, we may write:

$$\frac{1}{\pi}\left(H^-_T, H^+_T\right) \overset{(d)}{=} \left(\gamma_{g_{\sigma_1}}; \gamma_{\sigma_1} - \gamma_{g_{\sigma_1}}\right)$$

where $(\gamma_u, u \geq 0)$ is a Brownian motion independent from B, with respect to which σ_1, and g are defined.



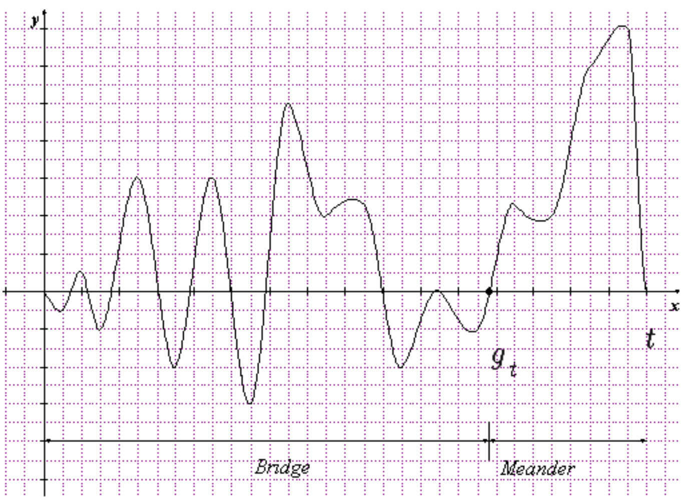
b) A particular case of the Feynman–Kac formula. As we know (from Lévy (1939)) that: $A_1^+ \stackrel{(d)}{=} \beta_{1/2,1/2}$, we obtain, by scaling:

$$A_T^{(+)} \stackrel{(d)}{=} \beta_{1/2,1/2}\bullet(2\mathfrak{e}) \stackrel{(d)}{=} 2\gamma_{1/2} \stackrel{(d)}{=} N^2$$

Thus, we find:

$$E[\exp(-\lambda A_T^{(+)})] = \frac{1}{(1+2\lambda)^{1/2}} \equiv \frac{1}{2}\int_0^\infty dt e^{-t/2} E[e^{-\lambda A_t^{(+)}}]$$

Explanation: above, T denotes an $\exp(1/2)$ variable; i.e: $P(T \in dt) = \frac{1}{2}e^{-t/2}dt$ which is independent of BM. ($T \equiv 2\mathfrak{e}$).
Then: $|B_T| \stackrel{(d)}{=} S_T \stackrel{(d)}{=} L_T \stackrel{(d)}{=} \mathfrak{e}$ ($\to$ Scaling + Duplication) and $|B_T|$ and L_T are independent because $(B_u, u \leq g_T)$ and $(B_{g_T+v}; v \leq T - g_T)$ are independent.
Why? $(B_u, u \leq g_t)$ and $(B_{g_t+v}; v \leq t - g_t)$ are conditionally independent given g_t and, thanks to the beta-gamma algebra, g_T and $(T - g_T)$ are independent;



(A.4) *Tsirel'son's equation.* In 1974, A. Zvonkin showed that the equation:

$$X_t = B_t + \int_0^t dsb(X_s)\ , \tag{2}$$

with b Bounded, Borel, admits a unique strong solution. (Note that here, b is not necessarily Lipschitz!!).

(Question: (Shiryaev)) Is it also true if the drift may depend upon the whole past of X? That is: $b(X_s) \longleftrightarrow T(s, X_\bullet)$, for some $T(s, X_\bullet)$ past-dependent.

B. Tsirel'son produced a counter example by exhibiting:

$$T(s, X_\bullet) = \sum_{k=-\infty}^{0} \left\{ \frac{X_{t_k} - X_{t_{k-1}}}{t_k - t_{k-1}} \right\} 1_{]t_k, t_{k+1}]}^{(s)}$$

$(t_k \downarrow 0, \text{as } k \downarrow -\infty)$.

The equation, or rather, its skeleton, is:

$$\eta_k \equiv \frac{X_{t_{k+1}} - X_{t_k}}{t_{k+1} - t_k} = \xi_k + \{\eta_{k-1}\}\ ,$$

with: $\xi_k = \left(\frac{B_{t_{k+1}} - B_{t_k}}{t_{k+1} - t_k} \right)$.

Then, one shows, simply enough:

$$\forall p \in \mathbb{Z} \diagdown \{0\}\ ,\ E[\exp(2i\pi p\{\eta_k\})|\mathcal{B}] = 0$$

i.e: $\{\eta_k\}$ is uniform, and independent from B.

B-Dressing up

(B.1)* Bringing in a BM to prove a result involving 1 or 2 r.v's. Neveu's proof of the hypercontractivity of the OU semigroup. See Neveu's paper [15].

(B.2) Calling upon a second BM to help a BM: *The Ciesielski–Taylor identities.* In 1962, Z. Ciesielski and S.J. Taylor [5] proved the remarkable identities in law:

$$\int_0^\infty ds 1_{(|B_s^{(n+2)}| \le 1)} \stackrel{\text{(law)}}{=} T_1(|B^n|)$$

by showing the identity of their Laplace transforms.

After rewriting both integrals in terms of the local times of these euclidian norm processes, so called Bessel processes, and using the adequate RK theorems, this identity in law boils down to (or may be understood as a particular case of):

$$\int_a^b -df(x) B^2_{g(x)} + f(b) B^2_{g(b)} \stackrel{\text{(d)}}{=} g(a) B^2_{f(a)} + \int_a^b dg(x) B^2_{f(x)} \tag{3}$$

where: $f, g : [a, b] \to \mathbb{R}$; $f \downarrow$; $g \uparrow$ continuous, a formula which resembles the integration by parts formula; in fact, it "contains" infinitely many such formulae... For simplicity, take f and g such that: $f(b) = g(a) = 0$ and f, g are C^1. Then, as a consequence of (3), we get:

$$\int_a^b -f'(x) dx B^2_{g(x)} \stackrel{\text{(d)}}{=} \int_a^b g'(x) dx B^2_{f(x)} \tag{†}$$

In order to prove this identity in law, we may consider:

$$\int_a^b dx(-f'(x))B^2_{g(x)} \overset{\text{(d)}}{=} \int_a^b dx(-f'(x))\left(\int_a^x \sqrt{g'(y)}dB_y\right)^2$$

as being the conditional variance of the stochastic integral (where (C_x) is a second BM independent of B):

$$\begin{aligned}&\int_a^b dC_x\sqrt{-f'(x)}\left(\int_a^x \sqrt{g'(y)}dB_y\right)\\ &\quad = \int_a^b dB_y\sqrt{g'(y)}\int_y^b dC_x\sqrt{-f'(x)}\end{aligned}$$

which gives:

$$\int_a^b dy(g'(y))\left(\int_y^b dC_x\sqrt{-f'(x)}\right)^2 \overset{\text{(d)}}{=} \int_a^b dy(g'(y))C^2_{(f(y))}$$

hence the desired result (†) since B and C are identically distributed.

(B.3) Calling upon the Wiener sheet to help a BM.

– Malliavin's calculus of variations is entirely based upon the use of the two parameter process:

$$e^{-t/2}W_{(u,e^t)}\ ,\ u\geq 0\,,\, t\in\mathbb{R}\ ,$$

as an Ornstein–Uhlenbeck process in t, taking values in path space $C(\mathbb{R}_+,\mathbb{R})$:

$$t\to e^{-t/2}W_{(\bullet,e^t)}$$

– More modestly, D. Baker and I [1] gave a proof of the following result due to Carr, Ewald, Xiao (Dec. 2008): the process:

$$A_t = \frac{1}{t}\int_0^t ds\exp\left(B_s-\frac{s}{2}\right)$$

is increasing in the convex order ie: $t\to E[\psi(A_t)]\uparrow$ for ψ convex.

– A theorem of Kellerer asserts that, for this, it is necessary and sufficient that there exists a martingale (M_t) such that: $\forall t$ fixed, $A_t \overset{\text{(d)}}{=} M_t$.

Now:

$$A_t = \int_0^1 du\exp\left(B_{ut}-\frac{ut}{2}\right) \overset{(1.d)}{=} \int_0^1 du\exp\left(W_{u,t}-\frac{ut}{2}\right)$$

and the RHS is a $(\mathcal{W}_t)$ martingale, where:

(B.4) From the arc sine law to the (infinite dim.) Poisson Dirichlet laws.

When looking at

$$A_t^+ = \int_0^t ds 1_{(B_s>0)}\ ,$$

one may write: $A_t^+ = \sum_n \varepsilon_n V_n(t)$; where $V_1(t)>V_2(t)>...>V_n(t),...$ are the lengths of excursions arranged in decreasing order and $\varepsilon_n=1$ (for an excursion >0), $=0$ if not.

Then, the sequence $\{\varepsilon_n\}$ consists of iid Bernoulli, independent of the sequence $\{V_n(t)\}$, and we obtain / we may show the extension of Lévy's result that:

$$\frac{1}{t}(V_1(t), ..., V_n(t)) \stackrel{\text{(d)}}{=} \frac{1}{\tau_\ell}((V_1)(\tau_\ell), (V_2)(\tau_\ell), ...)$$

This joint-infinite dimensional law is called Poisson–Dirichlet law. It plays some important role in Species distributions, and has many applications, including in number theory. In fact, there is a 2 parameter family $PD(\alpha, \theta)$ with $0 < \alpha < 1$, and $\alpha = 1/2$ is related to BM. See Pitman–Yor [18].

(B.5)* *Behind the asymptotic studies of planar BM additive functionals, there lies a phantom world.* See, e.g., Chapter XIII of Revuz–Yor [**?**].

(B.6) *Calling upon Itô's PPP of excursions to understand BM.* Itô's PPP of excursions can be used very simply and powerfully for many purposes: Itô's genius has been to replace the complicated / erratic / paths of BM by infinitely many staircases, ie: Poisson processes. But, given a problem, one only manipulates then / encounters / one such Poisson process. So, computations are made easy. Linear ODE's are used instead of Sturm–Liouville equations. Thus, Itô's excursion measure $n(d\varepsilon)$, that is the charac. measure of Itô's PPP becomes very fundamental, just as Wiener measure.

In particular, one may write:

$$\begin{aligned} E_W\Big(\exp\Big(-\lambda \underbrace{\int_0^{\tau_\ell} ds f(B_s)}_{F_\ell}\Big)\Big) &= \exp\Big(-\ell \int n(d\varepsilon)\Big(1 - \exp -\lambda \int_0^{V(\varepsilon)} ds f(\varepsilon_s)\Big)\Big) \\ &\equiv \exp\Big(-\ell \int n_f(dx)(1 - \exp(-\lambda x))\Big)\,, \end{aligned}$$

i.e: $n_f(dx)$, the image of $n(d\varepsilon)$ by: $\varepsilon \to \int_0^{V(\varepsilon)} ds f(\varepsilon_s)$, is Lévy's measure for the subordinator $(F_\ell, \ell \geq 0)$, where: $F_\ell = \int_0^{\tau_\ell} ds f(B_s)$.

My interest in the Hilbert transform of Brownian local times

$$H_t \equiv \int_0^\infty \frac{da}{a}(L_t^a - L_t^{-a})$$

stemmed from the remark that: $\left(\frac{1}{\pi}H_{\tau_\ell}, \ell \geq 0\right)$ is a standard Cauchy process. This may be compared with: $\gamma_{\tau_\ell} \stackrel{\text{(d)}}{=} \gamma_{\sigma_\ell}$ Spitzer's rep. So, H_t has the "grand idea" that it looks like a BM, ie: it has the same trace as $(\gamma_u, u \geq 0)$ on the set of zeros of B: $\{\tau_\ell, \ell \geq 0\}$. Thus, what about $\dfrac{1}{\pi}H_{\tau_\ell}$, given τ_ℓ? Is it Gaussian? Not at all, and using Itô's measure of excursions, P. Biane and I obtained [4]:

$$E\Big[\exp\Big(i\frac{\lambda}{\pi}H_{\tau_\ell} - \frac{\mu^2}{2}\tau_\ell\Big)\Big] = \exp(-\ell\lambda \coth(\lambda/\mu))$$

which bears/entertains some deep relationship with Lévy's formula for the stochastic area of planar BM, see, e.g., Mansuy–Yor [13].

The Brownian story (re: Hilbert transform of local times) was developed further first by Fitzsimmons–Getoor [8], for symmetric Lévy processes, then by Bertoin [2] for general Lévy processes.

(B.7)* Calling upon Markovian predictors to help understand non-Markovian processes: Knight's prediction theory. See Knight's book [11].

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